$\mathfrak{sl}_n$-WEBS, CATEGORIFICATION AND KHOVANOV-ROZANSKY HOMOLOGIES

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ABSTRACT. In this paper we define an explicit basis for the $\mathfrak{sl}_n$-web algebra $H_n(\vec{k})$, the $\mathfrak{sl}_n$ generalization of Khovanov's arc algebra $H_2(m)$, using categorified q-skew Howe duality.

Our construction is a $\mathfrak{sl}_n$-web version of Hu and Mathas graded cellular basis and has two major applications: It gives rise to an explicit isomorphism between a certain idempotent truncation of a thick calculus cyclotomic KL-R algebra and $H_n(\vec{k})$ and it gives an explicit graded cellular basis of the 2-hom space between any two $\mathfrak{sl}_n$-webs u and v. We use this to give a (in principle) computable version of colored Khovanov-Rozansky's $\mathfrak{sl}_n$-link homology. The complex we define is purely combinatorial and can be realized in the ("thick" cyclotomic) KL-R setting and needs only F's. Moreover, we discuss some applications of our construction on the uncategorified level related to dual canonical bases of the $\mathfrak{sl}_n$-web space $W_n(\vec{k})$ and the MOY-calculus. Latter gives rise to a method to compute colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomials.

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Which leaves open the question of what my personal contribution to this paper is.

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1. INTRODUCTION

1.1.1. *The framework.* The so-called *arc algebra* $H_2(m)$ was introduced by Khovanov in his influential paper [35] in order to extend his celebrated categorification of the Jones polynomial [34] to tangles. The algebra realizes the homology of a tangle with $2m$ top boundary points and $2m'$ bottom boundary points as certain $H_2(m)–H_2(m')$-bimodules. His algebra consists of $\mathfrak{sl}_2$-cobordisms in the sense of Bar-Natan [2] and has a beautiful diagrammatic calculus.

In the same vein, the so-called $\mathfrak{sl}_3$*-web algebra* K_S, introduced by Mackaay, Pan and the author in [51], consists of $\mathfrak{sl}_3$-foams in the sense of Khovanov [33] and is related to the $\mathfrak{sl}_3$-version of Khovanov homology from [33]. Shortly after the definition of K_S, Mackaay introduced in [50] the $\mathfrak{sl}_n$*-version of the arc algebra*, denoted by $H_n(\vec{k})$. These algebras use the matrix factorization framework introduced to the field of link homologies by Khovanov and Rozansky [43]. We should note that, using recent results of Queffelec and Rose [61][1], $H_n(\vec{k})$ could also be described using $\mathfrak{sl}_n$-foams introduced to the field by Mackaay, Stošić and Vaz in [53].

These algebras can be seen as the underlying algebraic structure for 2-categories of cobordisms/foams/matrix factorizations in the sense that these 2-categories are *equivalent* to certain (bi)module categories of these algebras, see in the literature cited above for details.

Moreover, the work of Brundan and Stroppel on generalizations of the arc algebra, intensively studied in the series of papers [9], [10], [11], [12] and [13] (and additionally studied e.g. in [22], [36], [71] and [72]), suggested that these algebras, in addition to their relations to knot theory, also have an interesting underlying representation theoretical and combinatorial structure. After their influential work the study of these algebras was carried out in great detail, e.g. the type A_2 variant was studied in [51], [63], [64], [74] and [75] and the type A_n-web algebra in [50]. There is also a type D version of the arc algebra, see [25], [26] and [27], and a $\mathfrak{gl}(1|1)$ variant [67].

In this paper we consider the $\mathfrak{sl}_n$-web algebra $H_n(\vec{k})$ from both sides: We study its *combinatorial and representation theoretical structure* and discuss its relation to the $\mathfrak{sl}_n$*-link polynomials/link homologies*. And, although we restrict ourself to $\bar{\mathbb{Q}}$, everything should work over $\mathbb{Z}$ *as well*.

1.1.2. *Relations and some history.* In order to get more precise let us recall that these algebras categorify the so-called $\mathfrak{sl}_n$*-web spaces* $W_n(\vec{k})$. These spaces consist of $\mathfrak{sl}_n$-webs who give a diagrammatic presentation of the representation category of $\mathbf{U}_q(\mathfrak{sl}_n)$. In the case $n=2$ this is well-known and already appeared in work of Rumer, Teller and Weyl [66] (in the non-quantum setting of course). For $n=3$ the diagrammatic calculus was introduced by Kuperberg in [44], but it was only recently proven in the $n>3$ case by Cautis, Kamnitzer and Morrison in [20], using q-skew Howe duality, that the $\mathfrak{sl}_n$-webs give rise to a diagrammatic presentation of $\mathbf{Rep}(\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}}))$.

These $\mathfrak{sl}_n$-webs are also related to the *MOY-calculus*, introduced by Murakami, Ohtsuki and Yamada in [59]. Therefore, these $\mathfrak{sl}_n$-webs can also be used in the context of the colored (we always mean k-colored with $\Lambda^k\bar{\mathbb{Q}}^n$, i.e. colored with the fundamental $\mathbf{U}_q(\mathfrak{sl}_n)$-representations) *Reshetikhin-Turaev* $\mathfrak{sl}_n$*-link polynomials*. The un-colored polynomials were *categorified* by Khovanov and Rozansky [43] using the language of matrix factorizations. Later Wu [79] and independently Yonezawa [81] have categorified the colored version. Thus, the $\mathfrak{sl}_n$-web algebras $H_n(\vec{k})$ have a direct connection to (colored) $\mathfrak{sl}_n$-link polynomials and $\mathfrak{sl}_n$-link homologies.

[1]Their results become available shortly after the first pre-print of this paper. But everything stated in this paper is also true using $\mathfrak{sl}_n$-foams instead of matrix factorizations.

It is worth noting that matrix factorizations are not the only way to define the $\mathfrak{sl}_n$-link homologies. In fact, there are many, e.g. using $\mathfrak{sl}_n$-foams [53], there is an approach using category $\mathcal{O}$, see [57], [70] and [73], while another approach is using derived categories associated to certain projective varieties, see [17] and [18]. Cautis and Kamnitzer's $\mathfrak{sl}_n$-link homologies are related to constructions by Manolescu [56] and Seidel and Smith [68] via mirror symmetry. And there is a version for $n = 2, 3$ by Lauda, Queffelec and Rose [45] that uses q-skew Howe duality and "higher" representation theory of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$. Moreover, the approach of Webster, recently updated in [77], to categorify the Reshetikhin-Turaev $\mathfrak{g}$-polynomial for arbitrary simple Lie algebra $\mathfrak{g}$, is another example. There are good reasons to believe that these are closely related. In fact, Webster conjectures in [77] (e.g. in the last paragraph) that all these link homologies should be the same. A first step for $n = 2, 3$ was done recently by Lauda, Queffelec and Rose [45] using categorified q-skew Howe duality and Chuang-Rouquier's version of the Rickard complex (see [23]).

But in all cases, including Khovanov and Rozansky's approach, calculations seem to be (very) hard for $n > 3$, see [14] and [62] for some calculations approaches. Moreover, the calculations in the $n = 2$, see [1], and $n = 3$, see [47], cases are based on the $\mathfrak{sl}_2$-cobordism or $\mathfrak{sl}_3$-foam framework respectively, where it is known for some time (see [54]) that the matrix factorization and the $\mathfrak{sl}_2$-cobordism or $\mathfrak{sl}_3$-foam approach give the same result. Note that a very recent approach to calculate the colored homology for rational tangles using categorified q-skew Howe duality was done by Wedrich in [78].

1.1.3. *Our motivation and approach.* Our approach is to obtain the Khovanov-Rozansky $\mathfrak{sl}_n$-link homologies using *(thick) cyclotomic KL-R algebras* and *categorified q-skew Howe duality*. Since these algebras have *an explicit basis*, one can write down the differentials explicitly with respect to these bases. Moreover, our complex is completely combinatorial in nature: *Neither* the matrix factorization framework *nor* $\mathfrak{sl}_n$-foams are needed.

Our motivation originated from the viewpoint of the combinatorial and representation theoretical structure of the $\mathfrak{sl}_n$-web algebra $H_n(\vec{k})$. To be more precise, it is known that the $\mathfrak{sl}_n$-web algebras are *graded cellular algebras* for any $n > 1$, see [51] and [55]. But *only an explicit* graded cellular basis would make it (in principle) possible to write down the set of graded, projective indecomposables which, under the identification mentioned above, correspond to indecomposable $\mathfrak{sl}_n$-web modules that categorify the dual canonical basis of $W_n(\vec{k})$.

But only in the $n = 2$ case there was a construction of an *explicit* graded cellular basis by Brundan and Stroppel [9]. That was the reason why the author in [75] used categorified q-skew Howe duality, loosely called $\mathfrak{sl}_3$-*foamation*, to define an *explicit* graded cellular basis of the $\mathfrak{sl}_3$-web algebra by giving a "foamy" version of Hu and Mathas [30] graded cellular basis of the *cyclotomic KL-R algebra* R_Λ (see Khovanov-Lauda [38], [39] or Rouquier [65]), where Λ denotes a dominant $\mathfrak{sl}_m$-weight. Note that Hu and Mathas results highly depend on Dipper, James and Mathas standard basis [24] of the cyclotomic Hecke algebra (which is graded isomorphic to R_Λ, see [6]).

It is worth noting that the construction in [75] can be easily adopted to the $\mathfrak{sl}_2$-cobordism framework using the $\mathfrak{sl}_2$-*foamation* of Lauda, Queffelec and Rose [45] (and Blanchet cobordisms [5] due to sign issues). Moreover, it turns out that the relation between $\mathfrak{sl}_3$-webs and the multitableaux language is surprisingly useful to study for example dual canonical bases of the $\mathfrak{sl}_3$-web spaces.

Thus, the starting motivation of the author was to extend this explicit basis to the $\mathfrak{sl}_n$-web algebras ${}_vH_n(\vec{k})_u$ for *any* $\mathfrak{sl}_n$-webs $u, v \in W_n(\vec{k})$ and *any* $\vec{k}$ (with entries summing up to a multiple of n). In order to do so, we follow the approach already indicated for $n = 3$ in [75], i.e. the usage

of *categorified, diagrammatic quantum skew Howe duality* studied recently independently in the $\mathfrak{sl}_n$-web framework in for example [20], [45] and [51] (and later extended to all $n > 1$ in [55]).

1.1.4. *$\mathfrak{sl}_n$-webs, q-skew Howe duality and combinatorics.* Let Λ denote n-times the ℓ-th fundamental $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight. The point is now that the q-skew Howe duality realizes the $\mathfrak{sl}_n$-web space $W_n(\Lambda)$ as the $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-module of highest weight Λ. In Lemma 4.9 we show something stronger, i.e. we give an *explicit* way to write any $\mathfrak{sl}_n$-web $u \in W_n(\vec{k})$ as a ($\bar{\mathbb{Q}}(q)$-multiple of a) certain string of *only* $F_i^{(j)}$'s acting as elements of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ under q-skew Howe duality: $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m)$ suffices (in fact, all $\mathfrak{sl}_n$-web relations follow *only* from the Serre relations) and we can see the $\mathfrak{sl}_n$-web spaces $W_n(\vec{k})$ as instances of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-*highest weight theory*.

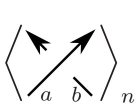
Using this explicit description in terms of $F_i^{(j)}$'s, it was not too hard to extend the neat relations between 3-multipartitions and $\mathfrak{sl}_3$-webs, 3-multitableaux and $\mathfrak{sl}_3$-flows and Brundan, Kleshchev and Wang's degree of 3-multitableaux (that comes from their work on graded Specht modules [8]) and weights of $\mathfrak{sl}_3$-flows (as the authors has worked out in detail in [75]) to all $n > 3$.

Moreover, recall that the $\mathfrak{sl}_n$-webs $u \in W_n(\vec{k})$ diagrammatically represent the *invariant tensors* $\mathrm{Inv}_{\mathbf{U}_q(\mathfrak{sl}_n)}(\bigotimes_i \Lambda^{k_i}\bar{\mathbb{Q}}^n) \cong \hom_{\mathbf{U}_q(\mathfrak{sl}_n)}(\bar{\mathbb{Q}}^2, \bigotimes_i \Lambda^{k_i}\bar{\mathbb{Q}}^n)$ and the $\mathfrak{sl}_n$-flows and their weights are a combinatorial way to express these vectors *explicitly* in terms of the *elementary tensors*. Thus, since the n-multipartition and n-multitableaux framework comes naturally when working with some kind of Hecke algebras, one can *loosely* say that the Hecke algebra "knows" the $\mathfrak{sl}_n$-web framework:

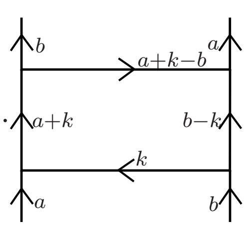
It is clear, using $\hom_{\mathbf{U}_q(\mathfrak{sl}_n)}(A,B) \cong \hom_{\mathbf{U}_q(\mathfrak{sl}_n)}(\bar{\mathbb{Q}}, A^* \otimes B)$ $(A, B \in \mathrm{Ob}(\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_n))))$ and the *bijection* between n-multitableaux and flows on $\mathfrak{sl}_n$-webs, explained in Section 4.1, that the $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiners can be explained *completely combinatorial* using (a version of) Specht theory.

Note now that for a closed $\mathfrak{sl}_n$-web w these $\mathfrak{sl}_n$-flows give the decomposition into elementary tensors of the trivial $\mathbf{U}_q(\mathfrak{sl}_n)$-representation $\bar{\mathbb{Q}}$, i.e. a certain quantum number. This number is the evaluation (up to a shift) of the $\mathfrak{sl}_n$-web w using the relations found in [20] - something that can not be done directly by an algorithm yet. But we state in Theorem 4.15 an *inductive evaluation algorithm* for *arbitrary* closed $\mathfrak{sl}_n$-webs by using only F's. Our algorithm uses the q-skew Howe duality and can be either stated in the combinatorial language of n-multitableaux (as we do) or in the algebraical language as the actions of the $F_i^{(j)}$'s of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ on a highest weight vector v_h. As an almost direct consequence we are able to prove an *explicit if-and-only-if condition* for a $\mathfrak{sl}_n$-web $u \in W_n(\vec{k})$ to be a dual canonical basis element, see Theorem 4.19.

We discuss another application of our algorithm in Section 4.2 : The evaluation of $\mathfrak{sl}_n$-webs is connected to *colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial* $\langle L_D \rangle_n$ (see e.g. [79]), but the usual translation of a a, b-colored crossing  into sums of $\mathfrak{sl}_n$-webs would use E's and F's, e.g.



$$= \sum_{k=0}^{b} \underbrace{(-1)^{k+(a+1)b} q^{-b+k}}_{\alpha(k)} \cdot$$

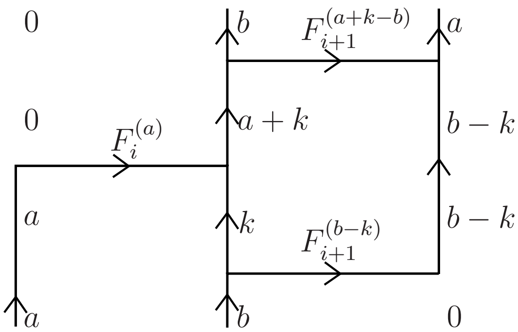


$$\rightsquigarrow \sum_{k=0}^{b} \alpha(k) \cdot F_i^{(a+k-b)} E_i^{(k)} v_{\dots,a,b\dots}$$

[2] By abuse of notation we always write $\bar{\mathbb{Q}}$ instead of $\bar{\mathbb{Q}}(q)$.

Thus, we had to *rearrange* it (this corresponds to an embedding of $\dot{\mathbf{U}}_q(\mathfrak{sl}_i)$ into $\dot{\mathbf{U}}_q(\mathfrak{sl}_{i+1})$ and then use the relations in $\dot{\mathbf{U}}_q(\mathfrak{sl}_{i+1})$ to re-write $F_i^{(a+k-b)}E_i^{(k)}$ in $\dot{\mathbf{U}}_q(\mathfrak{sl}_{i+1})$), using the observation that any $\mathfrak{sl}_n$-web can be obtained by a string of $F_i^{(j)}$'s, to

$$\sum_{k=0}^{b}\alpha(k)\cdot$$



$$\rightsquigarrow \sum_{k=0}^{b}\alpha(k)\cdot F_{i+1}^{(a+k-b)}F_i^{(a)}F_{i+1}^{(b-k)}v_{\dots a,b\dots}.$$

A neat fact is that the invariance under the Reidemeister moves, as we sketch in the proof of Theorem 4.31, are then just *instances* of the higher quantum Serre relations (which can be found e.g. in Chapter 7 of Lusztig's book [49]).

We give using this, as we explain in the Section 4.2, an *explicit algorithm* to compute the colored $\mathfrak{sl}_n$-MOY graph polynomials $\langle\cdot\rangle_{\mathrm{MOY}}$, and thus, the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-polynomials.

Our version is *completely combinatorial* in nature and has the nice upshot that there is no conceptual difference between different n and between the un-colored and colored setting.

1.1.5. *Categorified q-skew Howe duality.* Categorified q-skew Howe duality in the $\mathfrak{sl}_n$ case means that there is a strong $\mathfrak{sl}_m$-2-representation $\Gamma_{m,n\ell,n}\colon \mathcal{U}(\mathfrak{sl}_m)\to\mathcal{W}_\Lambda^p$ of Khovanov and Lauda's [40] categorification of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$, that we denote by $\mathcal{U}(\mathfrak{sl}_m)$, to a certain category of matrix factorizations (see [55] Definition 9.1) equivalent to a (suitable) module category $\mathcal{W}_\Lambda^p$ of the $\mathfrak{sl}_n$-web algebra $H_n(\Lambda)$ (see [50] Definition 7.1). This functor was used in [50] to show that $H_n(\Lambda)$ is Morita equivalent to a certain block of the cyclotomic KL-R algebra R_Λ[3].

Roughly: On the categorified level the observations above allow us to extend the construction of the "foamy" version of Hu and Mathas graded cellular basis to the $\mathfrak{sl}_n$ setting. We do this by giving a *growth algorithm* for homomorphisms (modulo null-homotopic maps) of matrix factorizations in Definition 5.10. These form a graded cellular basis, see Theorem 5.20. The procedure is *explicit* and two immediate advantages are that the growth algorithm gives a basis of $\mathrm{HOM}_{\mathrm{nh}}(\widehat{u},\widehat{v})$ for any $u,v\in W_n(\vec{k})$ (here $\widehat{u},\widehat{v}$ are certain associated matrix factorizations) and computations can be done completely *locally* using the cyclotomic KL-R relations, see [38] or [30] for a list of these relations in terms of diagrams or multitableaux. Another direct advantage of using only the cyclotomic quotients is that everything is *finite dimensional* and can be done using *explicit bases*.

And, as before, our construction is *completely combinatorial* using cyclotomic KL-R diagrams and the underlying "higher Specht combinatorics" of the Hu and Mathas basis. That is, one does not really need the matrix factorization (or $\mathfrak{sl}_n$-foams). But to get a little bit more precise what this means we need to talk about *thick calculus*.

1.1.6. *Divided powers and extended graphical calculus.* A main difference between the $\mathfrak{sl}_n$-web setting and the categorified quantum groups $\mathcal{U}(\mathfrak{sl}_m)$ is that the first is *closer* to its Karoubi envelope. That is, it is possible to use divided powers in the $\mathfrak{sl}_n$-web setting, but not directly for $\mathcal{U}(\mathfrak{sl}_m)$. For $\mathcal{U}(\mathfrak{sl}_m)$ one has to go to a full 2-subcategory of the Karoubi envelope $\dot{\mathcal{U}}(\mathfrak{sl}_m)$, denoted by $\check{\mathcal{U}}(\mathfrak{sl}_m)$,

[3]We note that we follow [51], [55] and [75] with our notation for $\mathcal{U}(\mathfrak{sl}_m)$, $\Gamma_{m,n\ell,n}$ and R_Λ.

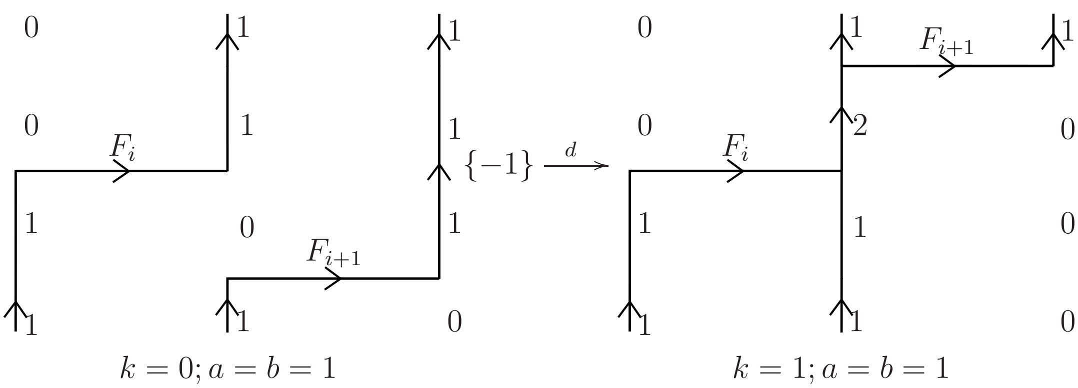
that we briefly recall in Section 3.3. Diagrammatically $\check{\mathcal{U}}(\mathfrak{sl}_m)$ is given by a version of extended graphical calculus, called *thick calculus*, from [42] where the reader can find more details.

In order to work with it, we extend Mackaay and Yonezawa's 2-functor to $\check{\mathcal{U}}(\mathfrak{sl}_m)$, see Theorem 3.36. Moreover, using Lemma 4.9 and Corollary 5.15, we show in Theorem 5.16 *explicitly* (by giving a *thick version* of the Hu and Mathas basis) that the extended 2-functor gives rise to an equivalence between the categories of modules over a certain block of the "thick" cyclotomic KL-R algebra, that we denote by $\check{R}_\Lambda$, and a suitable category of $H_n(\Lambda)$-modules.

In fact, we show in Theorem 5.16 that the $\mathfrak{sl}_n$-web algebra $H_n(\vec{k})$ is *isomorphic* to a (certain idempotent truncation) of $\check{R}_\Lambda$. Since $\check{R}_\Lambda$ can be studied completely combinatorial using "thick KL-R calculus" and the "thick" combinatorics of the Hu and Mathas basis, we can see this as a categorification of the corresponding results from the $\mathfrak{sl}_n$-web framework: Elements of $\mathrm{HOM}_{\mathrm{nh}}(\widehat{u},\widehat{v})$ are parametrized by pairs of n-multitableaux of a certain shape.

An interesting remark is that working with $\check{\mathcal{U}}(\mathfrak{sl}_m)$ (which is combinatorial not much more complicated than $\mathcal{U}(\mathfrak{sl}_m)$) suffices. That is, we can avoid working in the full Karoubi envelope $\dot{\mathcal{U}}(\mathfrak{sl}_m)$ where no diagrammatic or combinatorial definition is available for $n>2$ yet.

1.1.7. *$\mathfrak{sl}_n$-link homologies using combinatorics.* For the $\mathfrak{sl}_n$-link homologies this means that, using a complex as for example



with differential $d=\Gamma_{m,n\ell,n}(\times)\colon F_iF_{i+1}v_{\dots 1,1,0\dots}\{-1\}\to F_{i+1}F_iv_{\dots 1,1,0\dots}$, we can define a complex that *only* use the lower part $\mathcal{U}^-(\mathfrak{sl}_m)$. Since categorified q-skew Howe duality descents down to the cyclotomic KL-R algebra, we can define the complex using *only* the cyclotomic KL-R algebra with $d=\tilde{\Gamma}(\times)\colon F_iF_{i+1}v_{\dots 1,1,0\dots}\{-1\}\to F_{i+1}F_iv_{\dots 1,1,0\dots}$. Thus, we obtain in this way Khovanov-Rozansky's $\mathfrak{sl}_n$-link homology using "categorified" $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory.

The same works in the colored set-up using thick calculus and the (n-multitableaux combinatorics of the) thick cyclotomic KL-R algebra. And, as before for the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-polynomials, everything is *completely combinatorial* in nature and there is no conceptual difference between different n and between the un-colored and colored setting.

The *explicit* calculation of this complex is then a straightforward application of linear algebra: Use the $\mathfrak{sl}_n$-web version of the Hu and Mathas basis to write an explicit basis for all resolutions. The differential is then just given by applying a thick cyclotomic KL-R diagram from the left (stacking it on top) to the basis elements of the source. Then *pairing* the result with the *dual* of the thick Hu and Mathas basis for the target gives the differentials as a matrix. This gives an *explicit way* to compute the homology. It is worth noting again that for these calculations, due to

the local properties of the construction, the matrix factorizations framework is not really needed: The homology is governed by the combinatorics of the (thick) Hu-Mathas basis and the (thick) cyclotomic KL-R algebra. We explain how this works in Section 5.2.

1.1.8. *A note about foams.* The author was informed while typing this paper by Queffelec and Rose about their paper [61] where the authors have independently obtained similar results for the $\mathfrak{sl}_n$-link homologies (especially, they independently discovered that the $\mathfrak{sl}_n$-link homology can be obtained in the KL-R setting), but using $\mathfrak{sl}_n$-foams instead of matrix factorizations.

Note that Section 5.2, by similar arguments as in [45], can be extended to show that some of the aforementioned link homologies are the same. But this is not our purpose and is discussed in [61]. In fact, I like to thank Queffelec and Rose to point out to me that Chuang-Rouquier's version of the Rickard complex and the F-braiding complex I use (based on the observations above) are the same when passing to the (thick) Schur quotient (see [52] for the definition of the 2-Schur algebra).

Moreover, everything in this paper can be done with their $\mathfrak{sl}_n$-foams too, since the combinatorics of the (thick) cyclotomic KL-R and n-multitableaux suffices for everything. In fact, as before with the Serre relations on the uncategorified level, *all* the $\mathfrak{sl}_n$-foam relations are consequences of the (thick cyclotomic) KL-R relations. Although formally one would not need $\mathfrak{sl}_n$-foams: Some facts are easier to see using $\mathfrak{sl}_n$-foams (e.g. the isotopies) and others using n-multitableaux (e.g. the combinatorics). So we claim that both perspectives are worthwhile.

A neat fact about the $\mathfrak{sl}_n$-foam framework is that Brundan-Kleshchev-Wang's degree of multitableaux (which originated from their work on *graded*, higher Specht theory [8]) is, under the translation we discuss in Section 4.1 together with the $\mathfrak{sl}_n$-foamation of Queffelec and Rose and their Definition 3.3, then nothing else than a (slightly adjusted) *Euler characteristic* of foams.

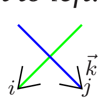
2. A short summary of the paper

2.1.2. *Summary of our notation.* We start by summarizing our notation to avoid confusion due to the fact that we are working in the overlap of different "worlds", i.e. the diagrammatic framework of $\mathcal{U}(\mathfrak{sl}_m)$ that consists of string diagrams, the combinatorial framework of the cyclotomic Hecke algebra that consists of multipartitions/multitableaux and the $\mathfrak{sl}_n$-web/matrix factorization framework that uses pictures (that is, the $\mathfrak{sl}_n$-webs) and the algebraic notion of matrix factorizations.

Since we tend to use highest and not lowest weight theory and F's and not E's, we think of a $\mathbf{U}_q(\mathfrak{sl}_2)$-representation V_N of highest weight N as

$$V_{-N} \underset{F}{\overset{E}{\rightleftarrows}} V_{-N+2} \underset{F}{\overset{E}{\rightleftarrows}} V_{-N+4} \underset{F}{\overset{E}{\rightleftarrows}} \dots \underset{F}{\overset{E}{\rightleftarrows}} V_{N-4} \underset{F}{\overset{E}{\rightleftarrows}} V_{N-2} \underset{F}{\overset{E}{\rightleftarrows}} V_N, \tag{2.1.1}$$

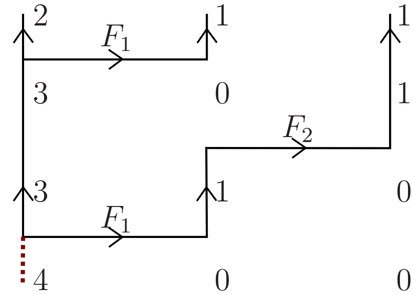
That is, we usually read from right to left. This is our reading convention for all diagrams of $\mathcal{U}(\mathfrak{sl}_m)$ and the cyclotomic KL-R algebra (thick ones as well): We think of them as being *a sequence of E's and F's ordered from right to left*. Moreover, we read them from bottom to top, i.e.

$\psi_3 =$  is $\psi_3 \colon \mathcal{F}_i\mathcal{F}_j\mathbf{1}_{\vec{k}} \Rightarrow \mathcal{F}_j\mathcal{F}_i\mathbf{1}_{\vec{k}}\{\alpha^{ij}\}.$

But since not many authors[4] seem to read $\mathfrak{sl}_n$-webs from right to left, we follow the standard of the majority. But to keep our notation a little bit consistent we read them in such a way that a turn of the diagrams by $\frac{\pi}{2}$ in clockwise direction matches the conventions before.

[4] The authors of [45] and [61] do. But they also use "lowest weight notation" by using E's. So I am safe here.

For example we read the string $F_1F_2F_1v_{(4,0,0)}$ as a $\mathfrak{sl}_4$-web (here the numbers of the grid correspond to the labels of the closed edges with the convention that we do not draw edges labeled 0 and the edges labeled n are pictured as a Bordeaux colored dotted line) as



Note that the labels of the middle and horizontal edges can easily be read of, since they are just the difference of the top right (left) and bottom right (left) numbers for the F's (the E's).

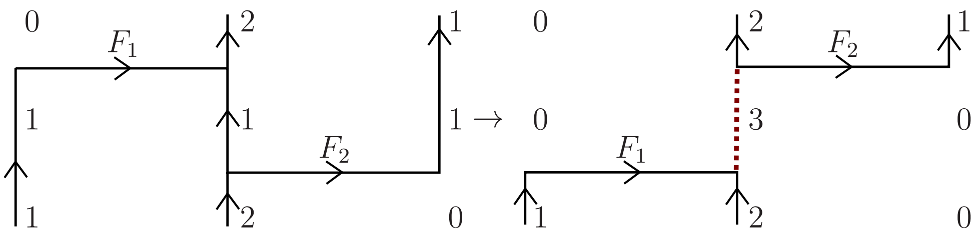
Thus, since we can see a $\mathfrak{sl}_n$-web u as a certain matrix factorization $\widehat{u}$ (see for example Section 5.4 in [55]), we can read a $\mathcal{U}(\mathfrak{sl}_m)$ diagram as a certain (equivalence class of) homomorphisms of matrix factorizations from the bottom $\mathfrak{sl}_n$-web u_b to the top $\mathfrak{sl}_n$-web u_t. Here the two $\mathfrak{sl}_n$-webs are obtained by letting the E's and F's for the bottom and top act on the weight vector $\vec{k}$.

We use the "highest weight notation" for the cyclotomic Hecke algebra too, i.e. reading multipartitions and multitableaux from right to left (the first entry is the *rightmost* etc.). Moreover, the elements of the $\mathfrak{sl}_n$-web algebra ${}_{u_t}H_n(\vec{k})_{u_b}$ are certain (equivalence classes of) homomorphisms of matrix factorizations $\mathcal{F}=\widehat{u}_b\to\widehat{u}_t$ that we inductively build from right to left. As an example: We decompose into

$$u_b=u_1\to u_2\to u_3\to\cdots\to u_{k-2}\to u_{k-1}\to u_k=u_t. \tag{2.1.2}$$

Then we use *stepwise* certain homomorphisms of matrix factorizations $\phi_i\colon\widehat{u}_i\to\widehat{u}_{i+1}$ and we set $\mathcal{F}=\phi_k\circ\cdots\circ\phi_1$. For example

$\widehat{CR}_{1,12}\colon$



is such a local step. Here $n=3$. The reader familiar with the $\mathfrak{sl}_2$ or $\mathfrak{sl}_3$ framework (see for example [35] or [51]) may think of it as building a $\mathfrak{sl}_2$-cobordism or $\mathfrak{sl}_3$-foam by composing (in a certain way) *basic pieces* such as saddles, zips, unzips and dotted identities. Roughly the same works for $\mathfrak{sl}_n$-foams and the reader can always think in terms of foams - if (s)he prefers foams.

2.1.3. *A rough sketch of our approach: The uncategorified world.* We start by giving a short summary of the relations between the three "worlds" mentioned above on the uncategorified level. A mnemonic diagram is

$$\begin{tikzcd}
n\text{-multitableaux} \arrow[r, leftrightarrow, "3.2 \text{ and } 4.1"] \arrow[rr, leftrightarrow, bend right, "\text{Section } 4.1"'] & \mathbf{Rep}(\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})) \arrow[r, leftrightarrow, "\text{Section } 3.2"] & \mathfrak{sl}_n\text{-webs}
\end{tikzcd}$$

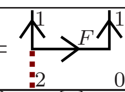
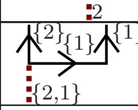
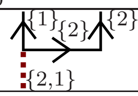
We call the three "worlds" loosely *combinatorics*, *representation theory* and *topology*. In our opinion all of them have their own advantages:

- For n-multitableaux everything is very explicit and can be done *inductively/algorithmically* by certain operations on n-multitableaux motivated by the classical story of the representation theory of the symmetric group (in fact, the cyclotomic Hecke algebra of level 1 is isomorphic to the group algebra of the symmetric group in the non-quantum setting).
- To study $\mathbf{Rep}(\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}}))$ is the category the author wants to *understand better*.
- The third is the category of $\mathfrak{sl}_n$-webs. Here it is easy to see the *"topology"*, e.g. isotopies and the connection to $\mathfrak{sl}_n$-link polynomials. In fact, it is non-trivial that the rather "rigid" n-multitableaux framework is isotopy invariant and on the other hand the $\mathfrak{sl}_n$-link polynomials are completely determined by this "rigid" combinatorics. This follows from the non-trivial translations in Sections 4.1 and 4.2.

Let us focus on $n = 2$ for now. The following table summarizes the uncategorified story.

Note that $\mathrm{hom}_{\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})}(\bar{\mathbb{Q}}, \bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}}^2) \cong \mathrm{Inv}_{\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})}(\bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}}^2) \subset \bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}}^2$. Fix for the $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$-vector representation $\bar{\mathbb{Q}}^2$ the basis $x_{\{1\}}$ and $x_{\{2\}}$ with the first vector in the $+1$- and the second in the -1-weight space of $\bar{\mathbb{Q}}^2$. We write $x_{12} = x_{\{1\}} \otimes x_{\{2\}}$ and $x_{21} = x_{\{2\}} \otimes x_{\{1\}}$.

The dotted line (leash) represents $\Lambda^2\bar{\mathbb{Q}}^2$, the empty space $\Lambda^0\bar{\mathbb{Q}}^2$ and the line $\Lambda^1\bar{\mathbb{Q}}^2 \cong \bar{\mathbb{Q}}^2$. Since the first two are trivial (the reader should think of the leash as non-existing: It encodes certain signs): The bottom/top of the right column "is" the source/target of the hom-space in the middle.

Combinatorics	Representation theory	Topology
$r((\,\emptyset\,,\,\boxed{1}\,)) = r((\,\boxed{1}\,,\,\emptyset\,))$	$u \in \mathrm{hom}_{\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})}(\bar{\mathbb{Q}}, \bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}}^2)$	$u =$  $\in W_2((1,1))$
$(\,\emptyset\,,\,\boxed{1}\,) \neq (\,\boxed{1}\,,\,\emptyset\,)$	$u = x_{21} - qx_{12} \in \bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}}^2$	 $\neq$ 
degree 0 and degree 1	coefficients q^0 and q^1	weight 0 and weight 1

To summarize: 2-multitableaux of the same residue sequence $r(\cdot)$ represent 1:1 certain $\mathfrak{sl}_2$-webs, 2-multitableaux represent 1:1 flows on these $\mathfrak{sl}_2$-webs and the degree of the 2-multitableaux gives the weight of the flows. It follows from the middle and the right columns that one can hope to get information about dual canonical bases[5] and about $\mathfrak{sl}_n$-link polynomials using the left column. This is essentially what we show in Section 4.1 and Section 4.2 respectively.

2.1.4. *A rough sketch of our approach: The categorified world.* From the categorified viewpoint one can hope that the n-multitableaux framework can be used to define cellular bases (since they give rise to a method to obtain the indecomposable modules that decategorify to the dual canonical basis) and an explicit method to obtain the $\mathfrak{sl}_n$-link homologies. This is essentially what we show in Section 5.1 and Section 5.2 respectively.

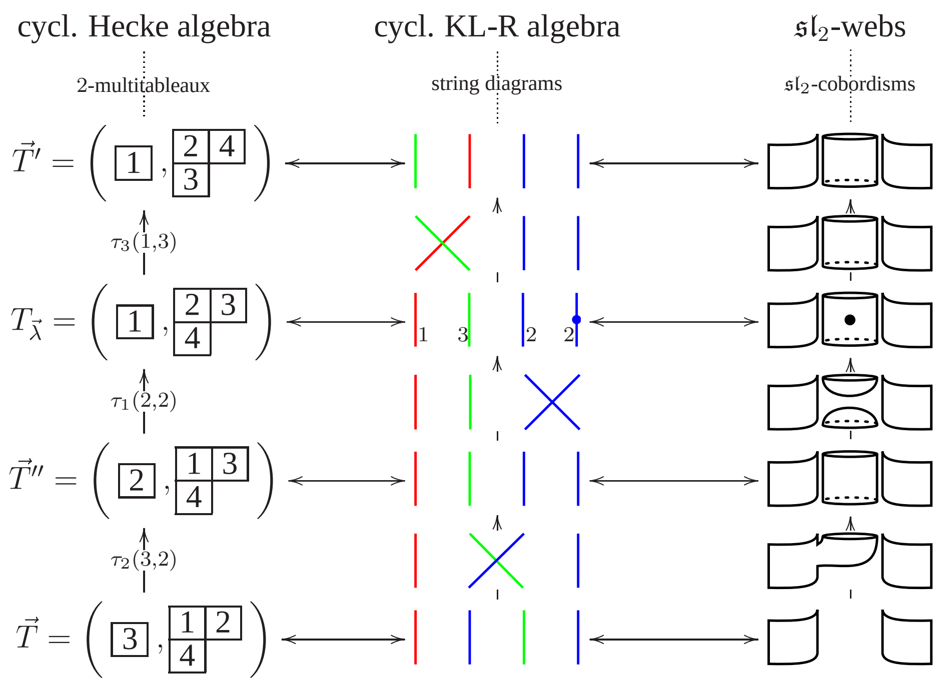
The crucial question is how to generate the string in 2.1.2. To do this we use (a thick) Hu and Mathas basis. This works roughly as follows. Fix two $\mathfrak{sl}_n$-webs $u, v \in W_n(\vec{k})$. There is a homogeneous $\bar{\mathbb{Q}}$-basis (that, even from the cyclotomic Hecke side, also works over $\mathbb{Z}$, see Theorem 3.14 in [31] or [48]) of $\mathrm{HOM}_{\mathrm{nh}}(\widehat{u}, \widehat{v})$ (or alternatively of $\mathfrak{sl}_n$-foam spaces) where each basis element is determinate by two n-multitableaux $\vec{T}, \vec{T}'$, one for u and one for v, with a certain fixed number

[5]For $\mathfrak{sl}_n$-webs a dual canonical basis in our notation is a "good basis for $q \to 0$" (positive exponent property).

of boxes $c(\vec{k})$. The string in 2.1.2 is generated by actions σ, σ' of elements of $S_{c(\vec{k})}$ by permuting nodes. The different basic pieces then depend on the difference of the residue of the permuted nodes. This can be seen as a "higher" analogon of classical Specht theory.

The actions are roughly obtained as follows. The n-multitableaux $\vec{T}, \vec{T}'$ are of the same shape, since the shape *only* depends on the boundary of the $\mathfrak{sl}_n$-webs. Then there is a n-multitableaux $T_{\vec{\lambda}}$ "in between" of the same shape with all its nodes filled in an ordered way. The actions are then given by applying a *suitable sequence of transpositions* $\tau_k(i,j), \tau'_k(i,j)$ from $T_{\vec{\lambda}}$ to $\vec{T}, \vec{T}'$.

Let us sketch in a diagram how the "higher" Specht basis roughly works. Here we focus on $n = 2$ and, as in Example 5.24, use Bar-Natan's cobordisms [2]. They are useful to illustrate the concepts, although one can not work with them due to sign issues, see [45]. In general one works with n-multitableaux, thick calculus and $\mathfrak{sl}_n$-matrix factorizations or $\mathfrak{sl}_n$-foams. Below we read again from bottom to top, i.e. the reader may think of the $\mathfrak{sl}_2$-web u sitting at the bottom and the $\mathfrak{sl}_2$-web v at the top (the colors in the middle column indicate the different residues of the nodes, e.g. $r(T_{\vec{\lambda}}) = (2,2,3,1)$). The element below is in $\hom_{\check{R}(\Lambda)}(F_1F_2F_3F_2, F_3F_1F_2F_2)$.



We stress again: Given $\tau_k(i,j)$, then one uses a certain $\mathfrak{sl}_2$-cobordism whose *position depends on* k and whose shape depends on the difference between the residues $|i-j|$. From bottom to top we see a saddle (difference 1), a cup-cap (difference 0) and a shift (difference > 1). The shift is hard to illustrate here but it just shifts the relative positions of the right and left arc, see also third diagram in Example 5.24. Another important fact is that all possible dots are *just* given by $T_{\vec{\lambda}}$. This corresponds to an identity with dots that *determines the cell* in the cellular basis.

We can use this "higher Specht basis" for the colored $\mathfrak{sl}_n$-link homologies as follows. In the language of Bar-Natan from [2]: The Khovanov chain complex has chain groups consisting of certain $\mathfrak{sl}_2$-webs and the differentials are $\mathfrak{sl}_2$-cobordisms between them. Thus, using the approach indicated above, we can formulate a chain complex whose *chain groups are strings of* $F_i^{(j)}$*'s* and

whose differentials are $\check{\mathcal{U}}(\mathfrak{sl}_m)$ *diagrams* between them. For example (compare to Example 5.43)

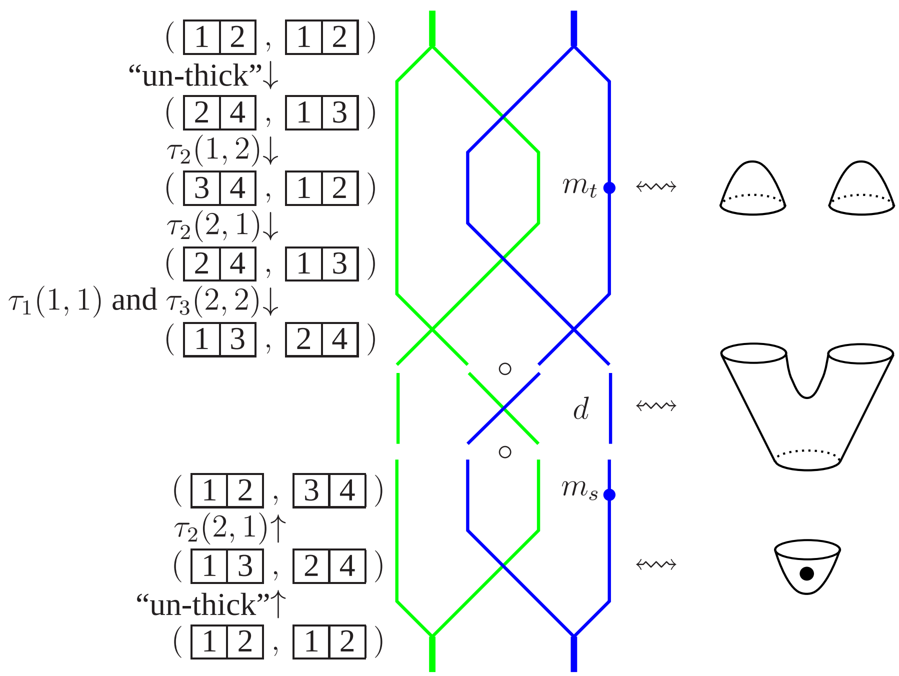
$$F_2F_1F_2F_1v_{(2^1)}\{-2\} \xrightarrow{\times\colon F_1F_2\rightarrow F_2F_1} F_2F_2F_1F_1v_{(2^1)}\{-1\}$$

would be such a complex. This can be thought of the local version of colored $\mathfrak{sl}_n$-link homology.

In Bar-Natan's picture: In order to do calculations one applies $\hom_{\check{R}(\Lambda)}(\emptyset,\cdot)$ and the chain groups are then given by (possibly dotted) cup's and the differential d is just given by gluing the $\mathfrak{sl}_2$-cobordism d on top of the cup's. Then use the dual (possibly dotted) cap basis of the target, evaluate the closed $\mathfrak{sl}_2$-cobordism and obtain numbers $\bar{\mathbb{Q}}$. This gives d as a matrix.

We do *literally the same*: We apply $\hom_{\check{R}(\Lambda)}(F^c_{(n^\ell)},\cdot)$ (where $F^c_{(n^\ell)}$ is a certain canonical string of leash-shifts that can be though of as non-existent). Now the chain groups are certain $\check{R}_\Lambda$-modules and the differentials are $\check{R}_\Lambda$-module maps given by composition from the right (gluing to the top).

The rest is also the same as in the Bar-Natan picture: Write a thick Hu and Mathas basis element m_s (the "cup basis") of the source, glue the differential d to its top and *pair* it with a thick Hu and Mathas *dual* basis element m_t (the "cap basis" which is literally obtained by reading everything backwards) of the target (here $F^c_{(n^\ell)} = F_2^{(2)}F_1^{(2)}$):



Above: The elements of the source are elements of the $\check{R}_\Lambda$-module $\hom_{\check{R}(\Lambda)}(F_2^{(2)}F_1^{(2)}, F_2F_1F_2F_1)$, the elements of the target are elements of the $\check{R}_\Lambda$-module $\hom_{\check{R}(\Lambda)}(F_2F_2F_1F_1, F_2^{(2)}F_1^{(2)})$ and the differential is a $\check{R}_\Lambda$-module map in $\hom_{\check{R}(\Lambda)}(F_2F_1F_2F_1, F_2F_2F_1F_1)$. Thus, the composite is an element of the 1-dimensional $\check{R}_\Lambda$-module $\hom_{\check{R}(\Lambda)}(F_2^{(2)}F_1^{(2)}, F_2^{(2)}F_1^{(2)})$: It is just a number in $\bar{\mathbb{Q}}$. This can be seen as the *evaluation* of closed $\mathfrak{sl}_n$-foams that categorifies our algorithm to evaluate closed $\mathfrak{sl}_n$-webs. This number can be obtained *explicitly* by using rules from thick calculus (see [42] or [69]) that can also be stated directly in terms of n-multitableaux. In fact, one can (if one likes) say that the evaluation of closed $\mathfrak{sl}_n$-foams is already inside of at least work by Hu and Mathas. Although the combinatorics go back even further, see the references in Section 6 of [30].

2.1.5. *A calculation example.* We sketch by an example our approach how to calculate the (colored) Khovanov-Rozansky $\mathfrak{sl}_n$-link homologies. We want to stress three things again before we start: The possibility for calculations is just *one application* of our translation. Moreover, it follows from Rouquier's universality Theorem (Corollary 5.7 in [65]) that all link homologies using the MOY-calculus as underlying uncategorified framework and analoga of Khovanov's original differentials have to give the *same* result (very, very roughly: The $\mathfrak{sl}_n$-web space $W_n(\Lambda)$ is *the* $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-representation of highest weight Λ and there is only *"one"* categorification of this). Thus, we do not need neither matrix factorizations nor $\mathfrak{sl}_n$-foams (we need them to show that everything *works*). Another point we would like to add: Our framework has enough local properties to perform an analogue of Bar-Natan's *"divide and conquer" algorithm* from [1]. His local simplifications seems to correspond on our side to the *categorification of the higher quantum Serre relations* by Stošić, see Sections 4 and 5 in [69]. Life is short, but this paper is *not*: We only sketch how this should work in Remark 5.33.

Now the example: This is the Hopf link example that also appears in the Examples 4.33 and 5.40 where the reader can find the pictures. We set $n = 3, m = 6$ and we have colored the two positive crossings with the colors 1 (left component) and 2. The presentation via $F_i^{(j)}$'s is

$$\mathrm{Hopf} = F_4^{(3)}F_5^{(2)}F_3^{(2)}F_2^{(2)}F_1^{(2)}T_{2,1,3}T_{1,2,2}F_5F_4F_3F_1F_2^{(3)}v_{(3,3,0,0,0,0)}.$$

where the T's represent the braiding and the right and left strings of $F_i^{(j)}$'s (that we shortly denote by F_b and F_t) correspond to the bottom and top closure respectively. The local braid complex $T_{2,1,3}T_{1,2,2}\tilde{v} = T_{2,1,3}T_{1,2,2}v_{\dots,1,2,\dots}$ (that technically takes place in a Schur quotient of $\check{\mathcal{U}}(\mathfrak{sl}_6)$) is

$$\begin{tikzcd}[column sep=small]
 & F_4^{(2)}F_3^{(2)}F_2F_3\tilde{v}\{-1\} \arrow[dr, ": F_2F_3\to F_3F_2"] & \\
F_4F_3^{(2)}F_4F_2F_3\tilde{v}\{-2\} \arrow[ur, ": F_4F_3^{(2)}F_4\to F_4^{(2)}F_3^{(2)}"] \arrow[dr, ": F_2F_3\to F_3F_2"'] & \oplus & F_4^{(2)}F_3^{(2)}F_3F_2\tilde{v}\{0\} \\
 & F_4F_3^{(2)}F_4F_3F_2\tilde{v}\{-1\} \arrow[ur, "-: F_4F_3^{(2)}F_4\to F_4^{(2)}F_3^{(2)}"'] &
\end{tikzcd}$$

with leftmost part in homological degree zero. In the rightmost part we see $F_3^{(2)}F_3$ that is isomorphic (given by an *explicit* isomorphism) to $[3]F_3^{(3)}$ (this is a shorthand notation for a shifted direct sum) in $\check{\mathcal{U}}(\mathfrak{sl}_6)$, see Theorem 5.1.1 in [42]. By using one of Stošić's categorifications of the higher quantum Serre relations (Theorem 3 in [69]), we see that $F_3^{(2)}F_2F_3$ is (in the Schur quotient) isomorphic (given by an *explicit* isomorphism) to $[2]F_3^{(3)}F_2$. Using a Gauss elimination (induced differential $\tilde{d}$!) we see that the middle top and the non-top degree part of the rightmost component will cancel and the complex simplifies to (with $d =$: $F_2F_3 \to F_3F_2$ as before)

$$F_4F_3^{(2)}F_4F_2F_3\tilde{v}\{-2\} \xrightarrow{d} F_4F_3^{(2)}F_4F_3F_2\tilde{v}\{-1\} \xrightarrow{\tilde{d}} F_4^{(2)}F_3^{(3)}F_2\tilde{v}\{2\}.$$

We now close it with F_t, F_b. By using $\hom_{\check{R}}(F^c_{(3^2)}, \cdot)$ and calculate the Hu and Mathas basis for the *left two* $\check{R}$-modules and the dual for the *right two* $\check{R}$-modules, we get, using the approach sketched above, the two differentials as matrices. Thus, calculating the homology is just linear algebra.

2.1.6. *Paper structure.* Before we summarize the paper let us note that Section 3 is (mostly) introducing notation and can be *skipped* by reader who feels safe using the language of $\mathfrak{sl}_n$-webs and categorified quantum groups. We try to illustrate everything with *plenty of examples* to help the reader on his/her way through this (too?) long paper. One can always go back to Section 3 and look for the explicit definitions.

The summary of the uncategorified picture in this paper is as follows.

We start in Sections 3.1 and 3.2 by recalling some notions and fix notations, e.g. the notions of n-multitableaux and $\mathfrak{sl}_n$-webs. Most parts in those sections are known, but we have also included new results related to our framework, e.g. in Theorem 3.28 we show how the flows and their weights corresponds to the decomposition into elementary tensors (we think this should be known, but we were unable to find the result in the literature).

In Section 4.1, among other things, we give a detailed discussion of the relation between the $\mathfrak{sl}_n$-webs and the n-multitableaux language.

The combinatorial heart of Section 4.1 is the extended growth algorithm from Definition 4.5 that gives a bijection between $\mathfrak{sl}_n$-webs with flows and n-multitableaux (see Proposition 4.8). This bijection can be extended to match Brundan-Kleshchev-Wang's degree of n-multitableaux with weights of flows (see Proposition 4.12).

We use this to give an evaluation algorithm in Theorem 4.15 and its application to the dual canonical basis in Theorem 4.19. Note that Lemma 4.9 implies that all relations from [20] follow from the higher Serre relations (see e.g. in Chapter 7 in [49]).

Section 4.2 contains the application of the evaluation algorithm for calculations of the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-polynomials in detail. That is, after showing in Lemma 4.30 how links can be explicitly see as strings of $F_i^{(j)}$'s, we show in Theorem 4.31 how to use n-multitableaux to compute the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-polynomials. A neat fact (although we only sketch how it works): The invariance under the Reidemeister moves is a consequence of the higher Serre relations. Afterwards we give two explicit examples (see Subsection 4.2.4).

The summary of the categorified picture in this paper is as follows.

We start in Section 3.3 by recalling some notions and fix notations, e.g. the notions of $\mathcal{U}(\mathfrak{sl}_m)$ and $\check{\mathcal{U}}(\mathfrak{sl}_m)$, the (thick cyclotomic) KL-R algebra $\check{R}_\Lambda$ and matrix factorizations. Most parts in those sections are known, but we have also included new results related to our framework, e.g. "thick" categorical q-skew Howe duality, see Theorem 3.36.

In Section 5.1 we give the $\mathfrak{sl}_n$-web version of the HM-basis by a growth algorithm, see Definition 5.10 (for the dual HM-basis see Remark 5.21), and show that it is a graded cellular basis in Theorem 5.20. Moreover, we relate our construction to the thick KL-R algebra in Theorem 5.16.

And in the last section, i.e. Section 5.2, we define our version of the colored $\mathfrak{sl}_n$-link homology in Definition 5.36 and show in Theorem 5.37 that it agrees with the colored Khovanov-Rozansky $\mathfrak{sl}_n$-link homology. Moreover, we discuss some local properties related to the Rickard complex in Lemma 5.30. Afterwards we show (Definition 5.41 and Theorem 5.42) how to use the $\mathfrak{sl}_n$-web version of the HM-basis for calculations.

Note that this shows that the Khovanov-Rozansky $\mathfrak{sl}_n$-link homologies are *completely combinatorial* in nature. Thus, everything is "down to earth" and can be made explicit.

We note again that, in order to illustrate that everything is explicit, we give *numerous examples*. We hope these will help the reader to understand the sometimes very confusing combinatorics.

3.1. Combinatorics, (multi)partitions and (multi)tableaux. In this section we define/recall the combinatorial notions about multitableaux that we use in this paper. We keep our notation close to the one used in [75]. In fact, most of this section is copied from there with the adoption to the more general case.

Choose an arbitrary but fixed non-negative integer $m \geq 2$. Let

$$\Lambda(m,d) = \left\{ \lambda = (\lambda^1, \ldots, \lambda^m) \in \mathbb{N}^m \mid \sum_{j=1}^{m} \lambda^j = d \right\}$$

be the set of *compositions* of d of length m. By $\Lambda^+(m,d) \subset \Lambda(m,d)$ we denote the subset of *partitions*, i.e. all $\lambda \in \Lambda(m,d)$ such that

$$\lambda^1 \geq \lambda^2 \geq \ldots \geq \lambda^m \geq 0.$$

Let $\Lambda^{(+)}(m,d)_I \subset \Lambda^{(+)}(m,d)$ be the subset of compositions (or partitions) whose entries are all in $I \subset \mathbb{N}$. In particular, for some fixed $M \in \mathbb{N}$ we use $\Lambda^{(+)}(m,d)_M$ as a notation for

$$\Lambda^{(+)}(m,d)_M = \left\{ \lambda = (\lambda^1, \ldots, \lambda^m) \in \mathbb{N}^m \mid \sum_{j=1}^{m} \lambda^j = d,\ \lambda^j \in \{0, \ldots, M\} \right\}.$$

Here we use a notation that we will use sometimes in different contexts, i.e. a notation as $(+)$ should be read that we mean both versions - *with or without* the $+$.

Recall that we can associate to each $\lambda \in \Lambda^+(m,d)$ a *diagram for* λ

$$\lambda = \{(r,c) \mid 1 \leq c \leq \lambda^j, 0 \leq r \leq m, j = 1, \ldots, m\},$$

which we, by a slight abuse of notation, denote by the same symbol λ. The elements of a diagram are called *nodes* N. For example, if $\lambda = (3,2,1)$, that is $d = 6, m = 3$, then

$$\lambda = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ \\ \cline{1-1}\end{array}.$$

Hence, we use the English notation to denote our partitions/diagrams. We associate, by convention, all partitions $(0, \ldots, 0)$ of zero to the empty diagram $\emptyset$.

A *tableau* T *of shape* λ is a filling of λ with (possible repeating) numbers from a chosen, fixed set $\{1, \ldots, k\}$. Such a tableau T is said to be *semi-standard*, if its entries increase along its rows (weakly) and columns (strictly), and *column-strict*, if its entries increase along its columns (strictly) with no conditions on rows. For example

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \qquad T_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

The tableau T_1 is semi-standard, but T_2 is only column-strict. We denote the set of all *semi-standard tableaux of shape* λ by $\mathrm{Std}^s(\lambda)$ and the set of all *column-strict tableaux of shape* λ by $\mathrm{Col}(\lambda)$.

In the same vein, a n-*multipartition* $\vec{\lambda} \in \Lambda^+(m,d,n)$ of d with length m is a n-tuple of partitions $\vec{\lambda} = (\lambda_n, \ldots, \lambda_1)$. Each of its components $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^{|\lambda_i|})$ is of length $|\lambda_i|$ such that their total

length is m and their total sum is d. We can associate to each $\vec{\lambda} \in \Lambda^+(m,d,n)$ a *diagram for* $\vec{\lambda}$

$$\vec{\lambda} = \{(r,c,i) \mid 1 \leq c \leq \lambda_i^j,\ 0 \leq r \leq |\lambda_i|,\ i = n, \dots, 1,\ j = 0, \dots, |\lambda_i|\},$$

which we, again by a slight abuse of notation, denote by the same symbol $\vec{\lambda}$. For example, if we have $\vec{\lambda} = (\lambda_4 = (3,2,1), \lambda_3 = (0), \lambda_2 = (4), \lambda_1 = (3,1))$, that is $d = 14, m = 6$ and $n = 4$, then

$$\lambda = \left(\ydiagram{3,2,1}\ ,\ \emptyset\ ,\ \ydiagram{4}\ ,\ \ydiagram{3,1} \right).$$

As before, a n*-multitableau* $\vec{T}$ *of shape* $\vec{\lambda}$ is a filling of $\vec{\lambda}$ with (possible repeating) numbers from a chosen, fixed set $\{1, \dots, k\}$. Such a tableau $\vec{T}$ is said to be *standard*, if its entries increase along its rows and columns (both strictly) and all repeating numbers appear at most once in T_i of $\vec{T} = (T_n, \dots, T_1)$ and all nodes with the same number are of the same residue[6]. The *residue* of a node is defined below in Definition 3.1.

We denote the set of all *standard tableau* $\vec{T}$ *of shape* $\vec{\lambda}$ by $\mathrm{Std}(\vec{\lambda})$. We note that we do not need the notion of semi-standard or column-strict n-multitableaux and, on the other hand, do not need the notion of standard tableaux. If not otherwise stated all appearing n-multitableaux are assumed to be standard.

There are two natural embeddings $\iota_{n_1}^{n_2}, \kappa_{n_1}^{n_2} \colon \Lambda^+(m,d,n_1) \to \Lambda^+(m,d,n_2)$ for $n_2 \geq n_1$, i.e.

$$\iota_{n_1}^{n_2}(\vec{\lambda}) = (\underbrace{(0), \dots, (0)}_{n_2 - n_1}, \lambda_{n_1}, \dots, \lambda_1) \text{ and } \kappa_{n_1}^{n_2}(\vec{\lambda}) = (\lambda_{n_1}, \dots, \lambda_1, \underbrace{(0), \dots, (0)}_{n_2 - n_1}).$$

We always use the first one $\iota_{n_1}^{n_2}$, since the first one fits our other conventions. But we always think of $\iota_{n_1}^{n_2}(\vec{\lambda})$ as a n_1-multipartition $\vec{\lambda}$.

Definition 3.1. Let $\lambda \in \Lambda^+(m,d)$ be a partition. Then we associate to each node $N = (r,c) \in \lambda$ of λ a *residue* $r(N)$ by the rule $r(N) = c - r + \ell$ where ℓ is the number of non-zero entries of λ. It should be noted that we see ℓ as being fixed by λ, even if we speak later about addable or removable nodes. Moreover, the convention for the shift of the residue by ℓ is a normalization that ensures that the lowest residue for nodes is exactly 1.

If $\vec{\lambda}$ is a n-multipartition, then we can use the same notions for each of its nodes $N = (r,c,i)$. This time ℓ is the maximal number of non-zero entries of the components of $\vec{\lambda}$.

An *addable node* N *of residue* $r(N) = k$ is a node N that can be added to the diagram of λ such that the new diagram is still the diagram of a partition and the residue is $r(N) = k$. We denote the *set of addable nodes of residue* k *of* λ by $\mathsf{A}^k(\lambda)$.

Similar, a *removable node* N *of residue* $r(N) = k$ is a node that can be removed from the diagram of λ such that the new diagram is still the diagram of a partition and the residue of N is $r(N) = k$. We denote the *set of removable nodes of residue* k *of* λ by $\mathsf{R}^k(\lambda)$.

Again, we can use the same notions for a n-multipartition $\vec{\lambda} \in \Lambda^+(m,d,n)$.

Moreover, we say a node $N_1 = (r_1, c_1, i_1)$ of $\vec{\lambda} = (\lambda_i)_{i=n}^1$ comes *before/left of (or after/right of)* another node $N_2 = (r_2, c_2, i_2)$ of $\vec{\lambda}$, denoted by $N_1 \preceq N_2$ (or $N_1 \succeq N_2$), iff $i_1 > i_2$ or $i_1 = i_2$ and

[6]We warn the readers familiar with the "usual" notion of n-multitableaux that we use this slightly generalized definition because we want to use divided powers later.

$r_1 \leq r_2$ (or $i_1 < i_2$ or $i_1 = i_2$ and $r_1 \geq r_2$). We use the obvious definitions for the notions *strictly* before $\prec$ and *strictly* after $\succ$.

For a fixed node N, we denote the *set of addable nodes of λ before N* with the same residue $r(N) = k$ by $\mathsf{A}^{k\prec N}(\lambda)$ and we denote the *set of addable nodes of λ after N* with the same residue $r(N) = k$ by $\mathsf{A}^{k\succ N}(\lambda)$.

In the same vein, for a fixed node N, we denote the *set of removable nodes of λ before N* with the same residue $r(N) = k$ by $\mathsf{R}^{k\prec N}(\lambda)$ and we denote the *set of removable nodes of λ after N* with the same residue $r(N) = k$ by $\mathsf{R}^{k\succ N}(\lambda)$.

Example 3.2. Let $\vec{\lambda} = (\lambda_3, \lambda_2, \lambda_1)$ be the following 3-multipartition (we have $\ell = 3$).

$$\lambda_3 = \begin{ytableau} 3 & 4 & 5 & 6 \\ 2 & 3 \\ 1 \end{ytableau}\,, \quad \lambda_2 = \begin{ytableau} 3 & 4 \\ 2 \end{ytableau}\,, \quad \lambda_1 = \begin{ytableau} 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 \end{ytableau}\,.$$

We have filled the nodes of $\lambda_{3,2,1}$ with the corresponding residues. Note that the residue is constant along the diagonals.

The set of addable nodes of residue 4 for $\vec{\lambda}$ and the set of removable nodes of residue 2 for $\vec{\lambda}$ are given by

$$\lambda_3 = \begin{ytableau} \; & \; & \; & \; \\ \; & \; & \cdot \\ \; \end{ytableau}\,, \quad \lambda_2 = \begin{ytableau} \; & \; \\ \times \end{ytableau}\,, \quad \lambda_1 = \begin{ytableau} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \cdot \end{ytableau}\,,$$

where we have indicated the addable nodes with a $\cdot$ and the removable with a $\times$. The removable node is after/right of the left addable and before/left of the right addable node. Moreover, in the following we demonstrate all nodes strictly before $\prec$ and strictly after $\succ$ a fixed node marked $-$.

$$\lambda_3 = \begin{ytableau} \prec & \prec & \prec & \prec \\ \prec & \prec \\ \prec \end{ytableau}\,, \quad \lambda_2 = \begin{ytableau} \prec & \prec \\ \prec \end{ytableau}\,, \quad \lambda_1 = \begin{ytableau} \; & - & \; & \; \\ \succ & \succ & \succ & \succ \\ \succ & \succ & \succ \end{ytableau}\,.$$

Let us recall Brundan, Kleshchev and Wang's definition of the degree of a n-multitableau as it appears in the context of cyclotomic Hecke algebras, see for example [8]. The reader familiar with their paper should be careful that we have to change their definition slightly again to make sense of entries that appear more than once (this is related to the usage or non-usage of divided powers). A similar definition already appears in [75].

Definition 3.3. Let $\vec{T} \in \mathrm{Std}(\vec{\lambda})$ be a n-multitableau $\vec{T} = (T_n, \ldots, T_1)$ such that T_i is a standard tableau and the numbers are from a fixed set $\{1, \ldots, k\}$. Then we associate to $\vec{T}$ a *sequence of n-multitableaux* $(\vec{T}^j)$ for each $j \in \{0, 1, \ldots, k\}$ where $\vec{T}^j = (T_n^j, \ldots, T_1^j)$ and $T^j_{n,\ldots,1}$ is obtained from $T_{n,\ldots,1}$ by deleting all nodes with numbers *strictly* bigger than j.

Moreover, we associate to it a *sequence of n-multipartitions* $(\vec{\lambda}^j)$ by removing the entries of the nodes of $(\vec{T}^j)$.

Example 3.4. Given the following standard 4-multitableau

$$\vec{T} = \left(\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}\,, \begin{ytableau} 4 \end{ytableau}\,, \begin{ytableau} 1 & 2 \end{ytableau}\,, \begin{ytableau} 4 \end{ytableau} \right),$$

we obtain the following sequence. First note that, by definition, $\vec{T}^0 = (\emptyset, \emptyset, \emptyset, \emptyset)$ and $\vec{T}^4 = \vec{T}$. The intermediate 4-multitableaux are

$$\vec{T}^1 = \left(\boxed{1}\,,\ \emptyset\,,\ \boxed{1},\ \emptyset \right), \quad \vec{T}^2 = \left(\boxed{1\,|\,2}\,,\ \emptyset\,,\ \boxed{1\,|\,2}\,,\ \emptyset \right), \quad \vec{T}^3 = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}\,,\ \emptyset\,,\ \boxed{1\,|\,2}\,,\ \emptyset \right).$$

Definition 3.5. (**Brundan, Kleshchev and Wang: Degree of a n-multitableau**) Let $\vec{T} \in \mathrm{Std}(\vec{\lambda})$ be a (filled with numbers from $\{1, \dots, k\}$) n-multitableau $\vec{T} = (T_n, \dots, T_1)$ as in Definition 3.3. For $j \in \{1, \dots, k\}$ let N^j denote the set of all nodes that are filled with the number j and let $\vec{T}^j$ denote as before the n-multitableau obtained from $\vec{T}$ by removing all nodes with entries $> j$.

The ***degree of*** $\vec{T}^j$, denoted by $\deg(\vec{T}^j)$, is defined to be

$$\deg(\vec{T}^j) = |\mathsf{A}^{k \succ N}(\vec{T}^j)| - |\mathsf{R}^{k \succ N}(\vec{T}^j)| - a \quad \text{with} \quad a = \sum_{i=0}^{N^j - 1} i,$$

where we use the convention to count all nodes $N \in N^j$ with the same number const step by step starting from the leftmost (Think: Raise the numbers of these nodes from left to right by a small amount $\varepsilon \ll 1$ such that the node in entry i is filled with number $\mathrm{const} + i\varepsilon$ and do the same as for n-multitableaux without repeating numbers).

The ***degree*** of the n-multitableau $\vec{T} = (T_n, \dots, T_1)$, denoted by $\deg_{\mathrm{BKW}}(\vec{T})$, is then defined by

$$\deg_{\mathrm{BKW}}(\vec{T}) = \sum_{j=1}^{k} \deg(\vec{T}^j).$$

Example 3.6. All of the following four standard 4-multitableaux have degree zero.

$$\vec{T}_1 = \left(\emptyset\,,\ \emptyset\,,\ \emptyset\,,\ \boxed{1} \right)\,,\ \vec{T}_2 = \left(\emptyset\,,\ \emptyset\,,\ \boxed{1}\,,\ \boxed{1} \right)\,,$$
$$\vec{T}_3 = \left(\emptyset\,,\ \boxed{1}\,,\ \boxed{1}\,,\ \boxed{1} \right)\,,\ \vec{T}_4 = \left(\boxed{1}\,,\ \boxed{1}\,,\ \boxed{1}\,,\ \boxed{1} \right).$$

To see this, we note that in the first case there is no node after $\succ$ the unique node N^1. Hence, $\deg(\vec{T}_1) = 0$. In the second case we have to calculate two steps. In the first step, i.e.

$$\left(\emptyset\,,\ \emptyset\,,\ \boxed{1}\,,\ \boxed{\cdot} \right),$$

we count one addable node of the same residue which we have marked with a $\cdot$, but for the second step there is again no node after $\succ$ the last node anymore. Hence, $\deg(\vec{T}_2) = 0$, since we have to take the shift from Definition 3.5 into account. For the third case we have to calculate three steps, i.e. the first and the second are

$$\left(\emptyset\,,\ \boxed{1}\,,\ \boxed{\cdot}\,,\ \boxed{\cdot} \right) \quad \text{and} \quad \left(\emptyset\,,\ \boxed{1}\,,\ \boxed{1}\,,\ \boxed{\cdot} \right),$$

where we have again indicated the addable nodes of the same residue with a $\cdot$. The third step is as before. Hence, $\deg(\vec{T}_3) = 0$, because of the shift. The last case works similar with a shift by 6.

It should be noted that it is possible that the degree (total or local) is negative. For example the last step of

$$\vec{T}_5 = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 8 & 9 & \\ \cline{1-2} \end{array}\,,\ \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 10 & \\ \cline{1-1} 11 & \\ \cline{1-1} \end{array}\,,\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 9 & \\ \cline{1-2} 7 & & \\ \cline{1-1} \end{array} \right)$$

has no addable nodes after $\succ$ the node N^{11} with the same residue, but one removable, namely the node filled with the entry 7. Hence, $\deg(\vec{T}_5^{11}) = -1$. The total degree in this case is

$$\deg_{\mathrm{BKW}}(\vec{T}_5) = 1 + 0 + 0 + 0 + 1 + 0 + 0 + 1 + 0 + 1 - 1 = 3.$$

Definition 3.7. Let $\vec{\lambda} = (\lambda_n, \dots, \lambda_1)$ and $\vec{\mu} = (\mu_n, \dots, \mu_1)$ be n-multipartitions in $\Lambda^+(m, d, n)$. Recall that $\lambda_i = (\lambda_i^1, \dots, \lambda_i^{|\lambda_i|})$ and $\mu_i = (\mu_i^1, \dots, \mu_i^{|\mu_i|})$ for $i \in \{n, \dots, 1\}$.

We say $\vec{\mu}$ *dominates* $\vec{\lambda}$, denoted by $\vec{\lambda} \trianglelefteq \vec{\mu}$, if

$$\sum_{i'=1}^{i-1} |\lambda_{n+1-i'}| + \sum_{j=1}^{|\lambda_{n+1-i}|} \lambda_{n+1-i}^j \leq \sum_{i'=1}^{i-1} |\mu_{n+1-i'}| + \sum_{j=1}^{|\mu_{n+1-i}|} \mu_{n+1-i}^j$$

for all $1 \leq i \leq n$. We write $\vec{\lambda} \triangleleft \vec{\mu}$, if $\vec{\lambda} \trianglelefteq \vec{\mu}$ and $\vec{\lambda} \neq \vec{\mu}$. It is easy to check that $\trianglelefteq$ is a partial ordering of the set of all n-multipartitions $\Lambda^+(m, d, n)$, called the *dominance order*.

This order can be *extended* to n-multitableaux in the following way. Suppose we have two standard n-multitableaux $\vec{T}_1 \in \mathrm{Std}(\vec{\lambda})$ and $\vec{T}_2 \in \mathrm{Std}(\vec{\mu})$ filled with numbers from $\{1, \dots, k\}$. As in Definition 3.3, we denote the corresponding n-multipartitions after removing all nodes with entries higher than $j \in \{1, \dots, k\}$ by $\vec{\lambda}^j$ and $\vec{\mu}^j$. Then

$$\vec{T}_1 \trianglelefteq \vec{T}_2 \Longleftrightarrow \vec{\lambda}^j \trianglelefteq \vec{\mu}^j \quad \text{for all} \quad j \in \{1, \dots, k\}.$$

Given $\vec{\lambda} \in \Lambda^+(m, d, n)$, we can associate to it two *unique standard n-multitableaux* $T_{\vec{\lambda}} \in \mathrm{Std}(\vec{\lambda})$ *and* $T_{\vec{\lambda}}^* \in \mathrm{Std}(\vec{\lambda})$ with the property

$$\vec{T} \in \mathrm{Std}(\vec{\lambda}) \Rightarrow \vec{T} \trianglelefteq T_{\vec{\lambda}} \text{ and } \vec{T} \in \mathrm{Std}(\vec{\lambda}) \Rightarrow T_{\vec{\lambda}}^* \trianglelefteq \vec{T}.$$

Note that $T_{\vec{\lambda}}$ is easily seen to be the n-multitableau with all entries in order from top to bottom, filling up rows before columns, and left to right and its *dual* $T_{\vec{\lambda}}^*$ has its entries ordered also from top to bottom, but filling up columns before rows, and from right to left.

If we want to use the definitions above in the slightly more general setting with multiple entries, then we, by convention, use the same notions as above after re-numbering all nodes with the same number (inductively starting with the lowest) increasing from left to right and shifting all other entries by the corresponding number.

Example 3.8. Intuitively $\vec{T}_1 \triangleleft \vec{T}_2$ means the numbers in $\vec{T}_1$ appear "earlier to the right" than in $\vec{T}_2$. For example, given the 3-multipartition

$$\vec{\lambda} = \left(\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline \\ \hline \end{array}, \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right),$$

we see that

$$T_{\vec{\lambda}} = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 7 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right) \quad \text{and} \quad T_{\vec{\lambda}}^* = \left(\begin{array}{|c|c|}\hline 5 & 7 \\ \hline 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right)$$

The left will dominate all $\vec{T} \in \mathrm{Std}(\vec{\lambda})$. For example

$$\vec{T} = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 4 & 6 \\ \hline 7 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right)$$

will be dominated, since

$$\vec{T}^4 = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\,,\ \emptyset,\ \boxed{4} \right) \trianglelefteq T^4_{\vec{\lambda}} = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\,,\ \boxed{4}\,,\ \emptyset \right).$$

The dual on the other hand is dominated by all the others.

Example 3.9. In order to compare

$$\vec{T} = \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}\,,\ \boxed{2}\,,\ \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 \\ \cline{1-1} \end{array} \right)$$

to other 3-multitableaux we change it to

$$\vec{T} = \left(\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}\,,\ \boxed{3}\,,\ \begin{array}{|c|c|}\hline 2 & 6 \\ \hline 7 \\ \cline{1-1} \end{array} \right)$$

Definition 3.10. Let $\vec{T} \in \mathrm{Std}(\vec{\lambda})$ be a n-multitableau. The *residue sequence* of $\vec{T}$, denoted by $r(\vec{T})$, is the k-tuple whose $j \in \{1, \dots, k\}$ entry is the residues of the node with number j. Moreover, the *residue sequence* of a n-multipartition $\vec{\lambda}$, denoted by $r(\vec{\lambda})$, is defined to be $r(\vec{\lambda}) = r(T_{\vec{\lambda}})$.

If the n-multitableau $\vec{T}$ has multiple entries with label j and all of them are of the *same* residue, then we can easily adopt the same definition as above.

Let us point out that we do not need the notion of n-multicompositions due to the fact that a column-strict tableau with n-columns is the same as a n-multipartition as we recall shortly in Section 4.1 in the discussion about (dual) canonical bases.

3.2. The $\mathbf{U}_q(\mathfrak{sl}_n)$-spiders and the $\mathbf{U}_q(\mathfrak{sl}_n)$-web spaces.

3.2.1. *Definition of the* $\boldsymbol{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$*-spider.* In this section we are going to define the $\mathbf{U}_q(\mathfrak{sl}_n)$*-spider category or* $\mathfrak{sl}_n$*-web-category* $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_n))$. We follow the description in the paper of Cautis, Kamnitzer and Morrison [20] (the reader should be careful since Mackaay uses in [50] a slightly different convention). That is, we first define the notion of the *free* spider $\mathbf{Sp}_f(\mathbf{U}_q(\mathfrak{sl}_n))$ and we define the category $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_n))$ as a certain quotient of it.

Our convention for reading diagrams is from bottom to top and left to right. With *diagram* we mean oriented, planar graphs with labeled edges, where all vertices are either part of the boundary, 2-valent or 3-valent. We call the 2-valent vertices *tags*. The boundary in our case are lines at either the bottom or the top of the diagrams with a certain number of fixed points ordered from left to right. Moreover, in the whole section let the letters a, b, c, d and e denote elements of $\{0, \dots, n\}$ for some fixed $n > 1$. To avoid all possible technical difficulties we always work over $\bar{\mathbb{Q}}$.

Furthermore, we use the convention that $[a]$ denotes the *quantum integer* (with $[0] = 1$), $[a]!$ denotes the *quantum factorial*, that is

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}} = q^{a-1} + q^{a-3} + \cdots + q^{-a+1} + q^{-a+1} \text{ and } [a]! = [0][1]\cdots[a-1][a],$$

and

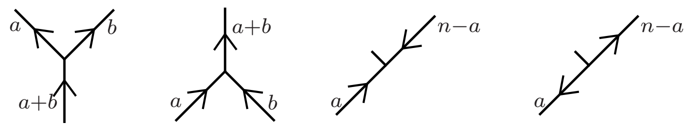
$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a]!}{[a-b]![b]!}$$

denotes the *quantum binomial*. We also use the convention $[-a] = -[a]$.

Definition 3.11. **(Free $\mathbf{U}_q(\mathfrak{sl}_n)$-spider)** Let $n > 1$. The *free $\boldsymbol{U}_q(\mathfrak{sl}_n)$-spider category*, which we denote by $\mathbf{Sp}_f(\mathbf{U}_q(\mathfrak{sl}_n))$, consists of the following data.

- The objects of $\mathbf{Sp}_f(\mathbf{U}_q(\mathfrak{sl}_n))$, denoted by $\mathrm{Ob}(\mathbf{Sp}_f(\mathbf{U}_q(\mathfrak{sl}_n)))$, are tuples $\vec{k}$ with entries in the set $\{0^{\pm},\dots,n^{\pm}\}$. We display their entries ordered from left to right according to their appearance in $\vec{k}$.
- The 1-morphisms of $\mathbf{Sp}_f(\mathbf{U}_q(\mathfrak{sl}_n))$ between $\vec{k}$ and $\vec{l}$, denoted by $\mathrm{Mor}_{\mathbf{Sp}_f(\mathbf{U}_q(\mathfrak{sl}_n))}(\vec{k},\vec{l})$, are diagrams between $\vec{k}$ and $\vec{l}$ freely generated as a $\bar{\mathbb{Q}}(q)$-vector space by all diagrams that can be obtained by gluing and juxtaposition of the following basic pieces (including the ones obtained by mirror reflections and arrow reversals).

(3.2.1)

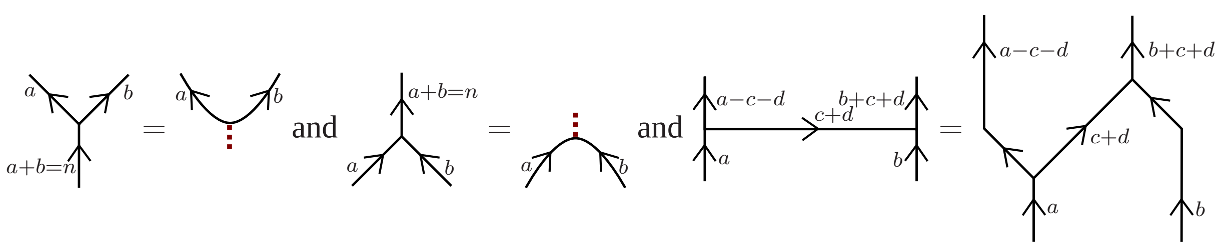


called (from left to right) *split (up)*, *merge (up)*, *tag in* and *tag out*. The tags have a distinguished site, i.e. they are not rotationally invariant. By convention, if the i-th bottom (top) boundary is a positive (negative) number, then the arrow is pointing out and vice versa for the other two possibilities. The boundary objects, by convention, should be the same as the label of the edge next to it iff the edge is pointing in and minus the label iff the edge is pointing out. Therefore, we usually do not picture the objects directly as e.g. in 3.2.1.

The category is $\bar{\mathbb{Q}}(q)$-linear, i.e. the spaces $\mathrm{Mor}_{\mathbf{Sp}_f(\mathbf{U}_q(\mathfrak{sl}_n))}(\vec{k},\vec{l})$ are $\bar{\mathbb{Q}}(q)$-vector spaces and composition is $\bar{\mathbb{Q}}(q)$-linear.

We usually do not draw edges labeled 0 and edges labeled n as dotted "leash" (see also Remark 3.20). We think of 0 and n-edges as *non-existing*. And, by convention, all diagrams with lower or bigger labels than 0 or n are defined to be 0.

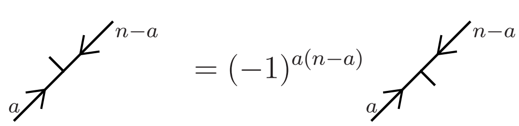
Moreover, we use shorthand notations for "ladders". It is worth noting that our at the first hand ambiguous way to draw these ladders is due to the fact that the bi-adjointness of E's and F's allows "isotopy relations" anyway. For example, we use the following diagrams (and similar ones for other "ladders") as a shorthand notation.



Definition 3.12. **($\mathbf{U}_q(\mathfrak{sl}_n)$-spider)** Let $n > 1$. The $\mathbf{U}_q(\mathfrak{sl}_n)$*-spider category*, which we denote by $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_n))$, is defined as a quotient of $\mathbf{Sp}_f(\mathbf{U}_q(\mathfrak{sl}_n))$, i.e. we take the quotient by some relations. The relation are the following plus mirror images and arrow reversals.

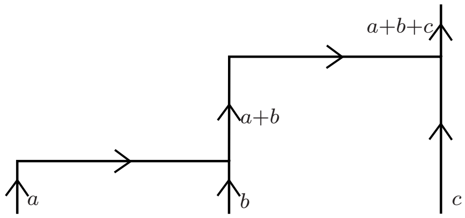
The relations split into three parts, i.e. the *tag* relation, the *"isotopy"* relations and the *"removal"* relations. The *tag relations* (recall that we include mirrors and arrow reflections) are

(3.2.2)

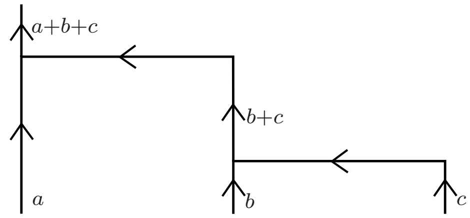


the *tripod relations*

(3.2.3)

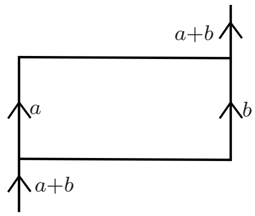


$=$

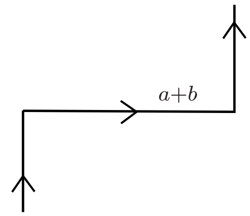


the *first digon removals*

(3.2.4)

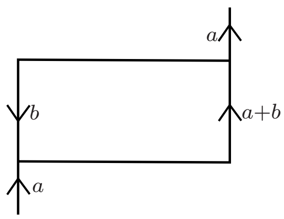


$= \begin{bmatrix} a+b \\ b \end{bmatrix}$

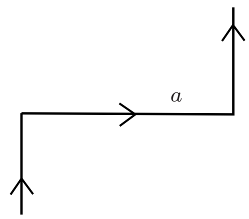


the *second digon removals*

(3.2.5)

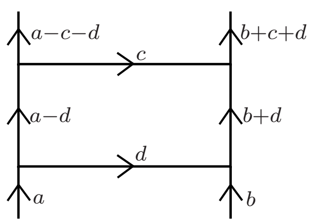


$= \begin{bmatrix} n-a \\ b \end{bmatrix}$

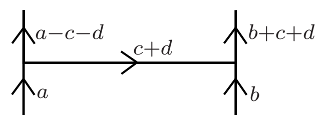


the *square removals*

(3.2.6)

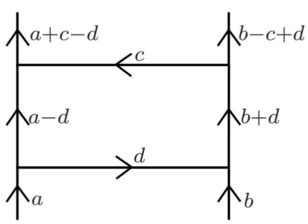


$= \begin{bmatrix} c+d \\ c \end{bmatrix}$

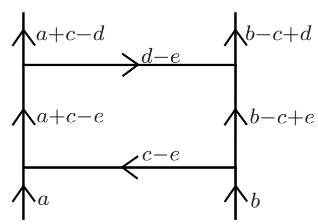


and the *square switches*

(3.2.7)



$= \sum_e \begin{bmatrix} a-b-c+d \\ e \end{bmatrix}$



Remark 3.13. We note the following.

(a) It follows from the second digon-removal 3.2.5 with $b = n-a$ that the tags are isomorphisms in **Sp**$(\mathbf{U}_q(\mathfrak{sl}_n))$. This allows us to consider only the full subcategory **Sp**$^+(\mathbf{U}_q(\mathfrak{sl}_n))$ consisting of just positive objects. Moreover, for any $0 \leq n' \leq n$ we can consider the full subcategory **Sp**$^{n'}(\mathbf{U}_q(\mathfrak{sl}_n))$ consisting of objects with labels in $\{0, \ldots, n'\}$.

(b) There are some useful relations that follow from the ones displayed above. But since we do not need them here, we just refer to the paper of Cautis, Kamnitzer and Morrison [20].

3.2.2. *Some* $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$*-representation theoretical notions.* Let us now briefly recall some of the representation theory of $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$. Much more details that are related to our framework can be found in [20] or [50]. We should note that the reader familiar with [20] or [50] should be careful with our notation, since we skip the subscript q in our notation. Moreover, since the quantized (for generic q) and the classical theory are very similar, we often use "$\mathfrak{sl}_n$-webs", "$\mathfrak{sl}_n$-weights" etc. instead of the longer versions "$\mathbf{U}_q(\mathfrak{sl}_n)$-webs", "$\mathbf{U}_q(\mathfrak{sl}_n)$-weights" etc.

First we recall the quantum general and special linear algebras. The $\mathfrak{gl}_n$-weight lattice is isomorphic to $\mathbb{Z}^n$. Let $\epsilon_i = (0,\dots,1,\dots,0) \in \mathbb{Z}^n$, with 1 being on the i-th coordinate, and $\alpha_i = \epsilon_i - \epsilon_{i+1} = (0,\dots,1,-1,\dots,0) \in \mathbb{Z}^n$, for $i = 1,\dots,n-1$. Recall that the Euclidean inner product on $\mathbb{Z}^n$ is defined by $(\epsilon_i,\epsilon_j) = \delta_{i,j}$. Moreover, a $\mathfrak{gl}_n$-weight $\vec{k} \in \mathbb{Z}^n$ uniquely determines a $\mathfrak{sl}_n$-weight $\vec{k} \in \mathbb{Z}^{n-1}$, which we, by abuse of notation, also denote $\vec{k}$, by

$$\vec{k} = (k_1,\dots,k_n) \in \mathbb{Z}^n \to \overline{k}(=\vec{k}) = (k_1-k_2,\dots,k_{n-1}-k_n) \in \mathbb{Z}^{n-1}. \tag{3.2.8}$$

Definition 3.14. For $n \in \mathbb{N}_{>1}$ the *quantum general linear algebra* $\mathbf{U}_q(\mathfrak{gl}_n)$ is the associative, unital $\bar{\mathbb{Q}}(q)$-algebra generated by K_i and K_i^{-1}, for $1,\dots,n$, and E_i, F_i (beware that some authors use E_{-i} instead of F_i, e.g. [50], [51] and [55]), for $i = 1,\dots,n-1$, subject to the relations

$$\begin{gathered}
K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,\\
E_iF_j - F_jE_i = \delta_{i,j}\frac{K_iK_{i+1}^{-1} - K_i^{-1}K_{i+1}}{q-q^{-1}},\\
K_iE_j = q^{(\epsilon_i,\alpha_j)}E_jK_i,\\
K_iF_j = q^{-(\epsilon_i,\alpha_j)}F_jK_i,\\
E_i^2E_j - [2]E_iE_jE_i + E_jE_i^2 = 0, \qquad \text{if} \quad |i-j| = 1,\\
E_iE_j - E_jE_i = 0, \qquad \text{else},\\
F_i^2F_j - [2]F_iF_jF_i + F_jF_i^2 = 0, \qquad \text{if} \quad |i-j| = 1,\\
F_iF_j - F_jF_i = 0, \qquad \text{else}.
\end{gathered}$$

Recall that the relations two and four read from the bottom are the so-called *Serre-relations*.

Definition 3.15. For $n \in \mathbb{N}_{>1}$ the *quantum special linear algebra* $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}}) \subseteq \mathbf{U}_q(\mathfrak{gl}_n)$ is the unital $\bar{\mathbb{Q}}(q)$-subalgebra generated by $K_i^{\pm 1}K_{i+1}^{\mp 1}$ and E_i, F_i, for $i = 1,\dots,n-1$.

It is worth noting that $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$ are Hopf algebras with coproduct Δ given by

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \;\; \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i \;\text{ and }\; \Delta(K_i) = K_i \otimes K_i.$$

The antipode S and the counit ε are given by

$$S(E_i) = -E_iK_i^{-1}, \;\; S(F_i) = -K_iF_i, \;\; S(K_i) = K_i^{-1}, \;\; \varepsilon(E_i) = \varepsilon(F_i) = 0 \;\text{ and }\; \varepsilon(K_i) = 1.$$

Recall that the Hopf algebra structure allows to extend actions to tensor products of representations, to duals of representations and there is a trivial representation.

Following Cautis, Kamnitzer and Morrison with our notation, we denote the standard basis of the $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$-representation $\bar{\mathbb{Q}}^n$ by $\{x_1, \ldots, x_n\}$, where the action is given by

$$E_i(x_j) = \begin{cases} x_{j-1}, & \text{if } i = j-1, \\ 0 & \text{else,} \end{cases} \quad F_i(x_j) = \begin{cases} x_{j+1}, & \text{if } i = j, \\ 0 & \text{else,} \end{cases} \quad K_i(x_j) = \begin{cases} qx_j, & \text{if } i = j, \\ q^{-1}x_j, & \text{if } i = j+1, \\ x_j & \text{else.} \end{cases}$$

Then we consider the following quotient of the tensor algebra $\mathcal{T}(\bar{\mathbb{Q}}^n)$ of $\bar{\mathbb{Q}}^n$

$$\Lambda^{\bullet}\bar{\mathbb{Q}}^n = \mathcal{T}(\bar{\mathbb{Q}}^n)/\mathcal{S}^2(\bar{\mathbb{Q}}^n),$$

where $\mathcal{S}^2(\bar{\mathbb{Q}}^n)$ is the symmetric square of $\bar{\mathbb{Q}}^n$ spanned by $x_i \wedge x_j + qx_j \wedge x_i$ for all pairs $i < j$ and by x_i^2 for all i. The interested reader can find details about the construction in the paper of Berenstein and Zwicknagl [4].

What is important for us now is that $\Lambda^{\bullet}(\bar{\mathbb{Q}}^n)$ is a graded algebra with product $\wedge$ and we denote by $\Lambda^k\bar{\mathbb{Q}}^n$ its k-th direct summand, that is

$$\Lambda^{\bullet}\bar{\mathbb{Q}}^n = \bigoplus_{k=0}^{n} \Lambda^k\bar{\mathbb{Q}}^n.$$

These summands are irreducible $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$-representations and the k-th one is called the *k-th fundamental* $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$-representation. We note that the $n-k$-th $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$-representation is isomorphic to the dual of the k-th one. Moreover, the two cases $k = 0, n$, which are duals, are called the *trivial* $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$-representation (recall that we denote it just by $\bar{\mathbb{Q}}$).

A notation that is important for us in the following is that, given an $>$-ordered k-element subset $S = \{s_1, \ldots, s_k\}$ of $\{n, \ldots, 1\}$[7], the *tensor basis* of $\Lambda^k\bar{\mathbb{Q}}^n$ is given by

$$\{x_S = x_{s_1} \wedge \cdots \wedge x_{s_k} \in \Lambda^k\bar{\mathbb{Q}}^n \mid S \subset \{n, \ldots, 1\}, |S| = k\}$$

and its elements are called *elementary tensors*. We need tensor products of these terms in the following. Therefore, as in [50], let $\vec{k} = (k_1, \ldots, k_m)$ be an m-tuple with $0 \leq k_i \leq n$ and define

$$\Lambda^{\vec{k}}\bar{\mathbb{Q}}^n = \Lambda^{k_1}\bar{\mathbb{Q}}^n \otimes \cdots \otimes \Lambda^{k_m}\bar{\mathbb{Q}}^n.$$

As Mackaay points out, the tensor basis can be extended to a basis of $\Lambda^{\vec{k}}\bar{\mathbb{Q}}^n$, which we, by abuse of notation, also call *tensor basis* and its elements $x_{\vec{S}}$ the *elementary tensors* of $\Lambda^{\vec{k}}\bar{\mathbb{Q}}^n$. Here we have $\vec{S} = (S_1, \ldots, S_m)$ with $S_j \subset \{n, \ldots, 1\}, |S_j| = k_j$ for $j = 1, \ldots, m$.

3.2.3. *Relation to the representation category* ***Rep***$(\boldsymbol{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}}))$. Before giving a more combinatorial descriptions, let us now briefly recall how the $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$-spider $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_n))$ is related to the representation category $\mathbf{Rep}(\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}}))$ of $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$. Recall that the objects of $\mathbf{Rep}(\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}}))$ are tensors of the $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$-representations $\Lambda^k\bar{\mathbb{Q}}^n, (\Lambda^k\bar{\mathbb{Q}}^n)^*$ and morphisms are intertwiners. Furthermore, recall that

[7] This is in fact a point of possible confusion. We follow Cautis, Kamnitzer and Morrison, i.e. the sets S are ordered decreasing. In order to make this visible, we write all involved sets decreasing.

the $\mathbf{U}_\mathrm{q}(\mathfrak{sl}_\mathrm{n})$-spider $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_n))$ is a so-called *pivotal* category: Roughly speaking it is monoidal with duals X^* such that $(X^*)^* \cong X$. The same holds for $\mathbf{Rep}(\mathbf{U}_\mathrm{q}(\mathfrak{sl}_\mathrm{n}))$.

An interesting fact is, since any finite dimensional, irreducible $\mathbf{U}_\mathrm{q}(\mathfrak{sl}_\mathrm{n})$-representation is a direct summand of a suitable tensor product of $\Lambda^k\bar{\mathbb{Q}}^n$'s, that the Karoubi envelope of $\mathbf{Rep}(\mathbf{U}_\mathrm{q}(\mathfrak{sl}_\mathrm{n}))$ is equivalent to the category of *all* finite dimensional $\mathbf{U}_\mathrm{q}(\mathfrak{sl}_\mathrm{n})$-representations.

Beware that we do not use $v = -q^{-1}$ as for instance Mackaay [50] in the following.

Given two subsets $S, T \subset \{n, \dots, 1\}$ define $\ell(S,T) = |\{(i,j) \in S \times T \mid i < j\}|$. For any $a, b \in \{1, \dots, n\}$ with $a + b \leq n$ define the following generating intertwiners.

(a) The intertwiner $M_s^{a,b}$ called *split* is given by

$$M_s^{a,b}\colon \Lambda^{a+b}\bar{\mathbb{Q}}^n \to \Lambda^a\bar{\mathbb{Q}}^n \otimes \Lambda^b\bar{\mathbb{Q}}^n,\ M_s^{a,b}(x_S) = \sum_{T\subset S}(-q)^{\ell(S,T)}x_T \otimes x_{S-T}.$$

(b) The intertwiner $M_m^{a,b}$ called *merge* is given by

$$M_m^{a,b}\colon \Lambda^a\bar{\mathbb{Q}}^n \otimes \Lambda^b\bar{\mathbb{Q}}^n \to \Lambda^{a+b}\bar{\mathbb{Q}}^n,\ M_s^{a,b}(x_S \otimes x_T) = \begin{cases}(-q)^{-\ell(T,S)}x_{S\cup T}, & \text{if } S \cap T = \emptyset,\\ 0 & \text{else.}\end{cases}$$

(c) The intertwiner D^a called *tag* is given by

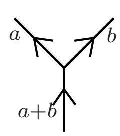
$$D^a\colon \Lambda^a\bar{\mathbb{Q}}^n \to (\Lambda^{n-a}\bar{\mathbb{Q}}^n)^*,\ D^a(x_S)(x_T) = \begin{cases}(-q)^{-\ell(S,T)}, & \text{if } S \cap T = \emptyset,\\ 0 & \text{else.}\end{cases}$$

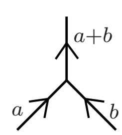
It is worth noting that the copairing and pairing (which belong to the cup and cap under the functor from Definition 3.16 below) are the special case $a + b = n$ of the split and merge given above.

Definition 3.16. (**Cautis-Kamnitzer-Morrison**) We define a pivotal functor, which we denote by $\Psi\colon \mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_n)) \to \mathbf{Rep}(\mathbf{U}_\mathrm{q}(\mathfrak{sl}_\mathrm{n}))$, given on objects by sending a $\vec{k} = (k_1^{\pm1}, \dots, k_m^{\pm1})$ with $k_i \in \{0, \dots, n\}$ to the corresponding tensor product of $\mathbf{U}_\mathrm{q}(\mathfrak{sl}_\mathrm{n})$-fundamental representations, i.e.

$$\vec{k} = (k_1^{\pm1}, \dots, k_m^{\pm1}) \mapsto (\Lambda^{k_1}\bar{\mathbb{Q}}^n)^{\pm1} \otimes \cdots \otimes (\Lambda^{k_m}\bar{\mathbb{Q}}^n)^{\pm1},$$

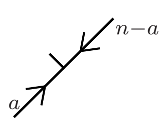
where a minus should indicate the dual $\mathbf{U}_\mathrm{q}(\mathfrak{sl}_\mathrm{n})$-representation. For the morphisms of $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_n))$ the functor Ψ is given by

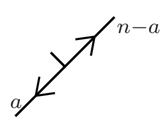
(3.2.9) 

$\mapsto M_s^{a+b}$ and 

$\mapsto M_m^{a+b}$

and

(3.2.10) 

$\mapsto D^a$ and 

$\mapsto (-1)^{a(n-a)}D^a$.

Theorem 3.17. ***(Cautis-Kamnitzer-Morrison** - Theorem 3.3.1 in* [20]*) The functor Ψ from above is a well-defined equivalence of pivotal categories **Sp**$(\mathbf{U}_q(\mathfrak{sl}_n))$ to **Rep**$(\mathbf{U}_\mathrm{q}(\mathfrak{sl}_\mathrm{n}))$.* $\square$

3.2.4. *Ladder moves and q-skew Howe duality.* We shortly recall q-skew Howe duality here. The main source for the reader interested in more details is the paper of Cautis, Kamnitzer and Morrison [20]. We start by recalling Beilinson-Lusztig-MacPherson [3] idempotent form of $\mathbf{U}_q(\mathfrak{sl}_n)$, denoted by $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$. It is worth noting that such an algebra can be seen as a 1-category and, as long as we only want to consider weight-representations, it contains the same amount of information and it eases to work with weight representations. Although it is a non-unital algebra.

Adjoin an idempotent $1_{\vec{k}}$ for $\mathbf{U}_q(\mathfrak{sl}_n)$ for each $\vec{k} \in \mathbb{Z}^{n-1}$ and add the relations

$$\begin{aligned} 1_{\vec{k}}1_{\vec{l}} &= \delta_{\vec{k},\vec{l}}1_{\vec{k}}, \\ E_i 1_{\vec{k}} &= 1_{\vec{k}+\overline{\alpha}_i}E_i, \quad \text{with } \overline{\alpha}_i \text{ as in Equation } 3.2.8, \\ F_i 1_{\vec{k}} &= 1_{\vec{k}-\overline{\alpha}_i}F_i, \quad \text{with } \overline{\alpha}_i \text{ as in Equation } 3.2.8, \\ K_iK_{i+1}^{-1}1_{\vec{k}} &= q^{\vec{k}_i}1_{\vec{k}}. \end{aligned}$$

Definition 3.18. The idempotented quantum special linear algebra is defined by

$$\dot{\mathbf{U}}_q(\mathfrak{sl}_n) = \bigoplus_{\vec{k},\vec{l}\in\mathbb{Z}^{n-1}} 1_{\vec{k}}\mathbf{U}_q(\mathfrak{sl}_n)1_{\vec{l}}.$$

The morphisms of the algebra (or 1-category) are generated for $i = 1, \ldots, n-1$ by the *divided powers*

$$E_i^{(j)} = \frac{E_i^j}{[j]!} \quad \text{and} \quad F_i^{(j)} = \frac{E_i^j}{[j]!}.$$

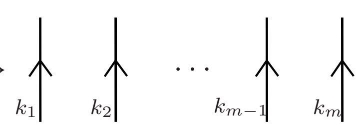
At this point it is worth noting that we try to carefully distinguish between weights $\vec{k}$ and compositions or tableaux λ. Although they can be thought to be similar in some sense, we use the language of compositions or tableaux for the combinatorics and the notion of weights for the representation theory.

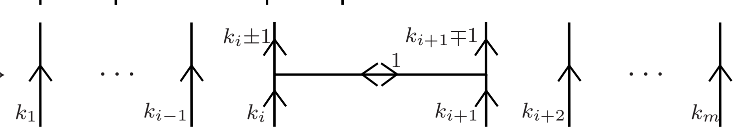
To define q-skew Howe duality on the level of $\mathfrak{sl}_n$-webs with m boundary points we restrict to certain weights $\vec{k}$ that we call n*-bounded* (see also Remark 3.13). These weights have only entries $0 \leq k_i \leq n$. Denote by a superscript n the subalgebras with only these weights. The following proposition is due to Cautis, Kamnitzer and Morrison. We call it *pictorial q-skew Howe duality*. How the functor is defined for the objects of $\dot{\mathbf{U}}_q^n(\mathfrak{sl}_m)$ can be found in [20] for instance. It should be noted that Cautis, Kamnitzer and Morrison describe in [20] q-skew Howe duality by an $\mathbf{U}_q(\mathfrak{gl}_m)$-action, while we are mostly using $\mathbf{U}_q(\mathfrak{sl}_m)$-actions with weights given by Equation 3.2.8.

Proposition 3.19. *(**Pictorial q-skew Howe duality** - Section 5 in* [20]*) The functor*

$$\gamma_{m,n}\colon \dot{\boldsymbol{U}}_q^n(\mathfrak{sl}_m) \to \boldsymbol{Sp}(\boldsymbol{U}_q(\mathfrak{sl}_n))$$

determined on morphisms by

$1_{\vec{k}} \mapsto$ 

$E_i1_{\vec{k}}, F_i1_{\vec{k}} \mapsto$ 

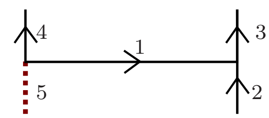
where the orientation of the arrow in the middle of the ladder is to the left for E and to the right for F, is well-defined, pivotal and full. This defines an $\mathbf{U}_q(\mathfrak{sl}_m)$-action on the $\mathfrak{sl}_n$-spider.

We note that the image of the divided powers is crucial and easy to write down, i.e. for $E_i^{(j)}$ and $F_i^{(j)}$ the middle arrow will have a label j and the two shifts at the top will also be by j instead of 1.

Remark 3.20. In order to work with the ladders in a pictorial convenient way we have to use the following convention, which we call *leash-convention*.

- Edges labeled 0 are not pictured.
- Edges labeled n are pictured using *dotted leashes* that we tend to picture as Bordeaux colored edges. We do not picture orientation for leashes, but it should be clear from the context.

This has the advantage that ladders corresponding to F's (the ones we mostly use) will always point upwards. An example with $n = 5$ is the following.



Note that the leashes keep track of the fact that a $\mathbf{U}_q(\mathfrak{sl}_m)$-representation and its dual are isomorphic, but the natural isomorphism induced by the antipode comes with a *sign*.

3.2.5. *The $\mathfrak{sl}_n$-web space.* Now we are going to define of the $\mathfrak{sl}_n$-*web space* and afterwards in Subsection 3.2.6 the $\mathfrak{sl}_n$-*flow lines* in the spirit of Khovanov and Kuperberg [37]. We note that we *only* use n-bounded $\vec{k}$, i.e. $k_i \in \{0, \dots, n\}$. By abuse of notation, we tend to suppress the "n-bounded" from our notation. Moreover, we write $(n^\ell) = (n, \dots, n, 0, \dots, 0) \in \Lambda(m, n\ell)_n$.

Definition 3.21. (**The $\mathfrak{sl}_n$-web space**) Given a fixed $\vec{k} \in \Lambda(m, n\ell)_n$ for some $\ell \in \mathbb{N}$, the $\mathfrak{sl}_n$-*web space for* $\vec{k}$, denoted by $W_n(\vec{k})$, is defined by

$$W_n(\vec{k}) = \mathrm{Mor}_{\mathbf{Sp}^n(\mathbf{U}_q(\mathfrak{sl}_n))}((n^\ell), \vec{k}) \cong \mathrm{Inv}_{\dot{\mathbf{U}}_q(\mathfrak{sl}_n)}(\Lambda^{\vec{k}}\bar{\mathbb{Q}}^n).$$

The $\mathfrak{sl}_n$-*web space* $W_n(\Lambda)$ (Λ denotes n-times the ℓ-th fundamental $\mathfrak{sl}_m$-weight) is defined by

$$W_n(\Lambda) = \bigoplus_{\vec{k} \in \Lambda(m, n\ell)_n} W_n(\vec{k}) = \bigoplus_{\vec{k} \in \Lambda(m, n\ell)_n} \mathrm{Mor}_{\mathbf{Sp}^n(\mathbf{U}_q(\mathfrak{sl}_n))}((n^\ell), \vec{k}).$$

Note that q-skew Howe duality gives $W_n(\Lambda)$ the structure of the irreducible $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-module with *highest* weight Λ (see [55] Corollary 4.10 for details).

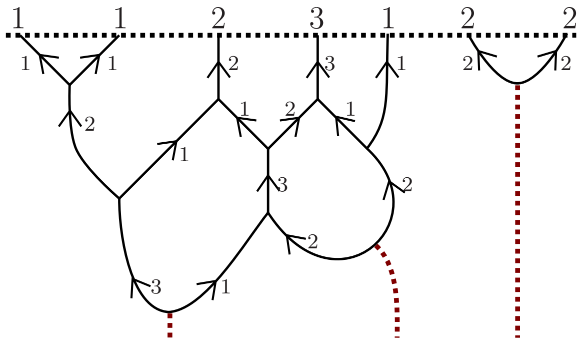
Moreover, due to the fact that

$$\mathrm{Mor}_{\mathcal{C}}(X, Y) \cong \mathrm{Mor}_{\mathcal{C}}(\mathbf{1}, X^* \otimes Y) \cong \mathrm{Mor}_{\mathcal{C}}(\mathbf{1}, Y \otimes X^*) \tag{3.2.11}$$

holds in any pivotal category $\mathcal{C}$ with identity $\mathbf{1}$, we note that the $\mathfrak{sl}_n$-web spaces are just a convenient way to work with the $\mathbf{U}_{\mathrm{q}}(\mathfrak{sl}_{\mathrm{n}})$-intertwiners.

Boundaries of $\mathfrak{sl}_n$-webs consist of univalent vertices (the end points of oriented edges), which we will usually put on a horizontal line (or various horizontal lines), called the *cut-line*, and that

we usually picture by a dotted line, e.g. such a $\mathfrak{sl}_n$-web is shown below for $n = 4$.

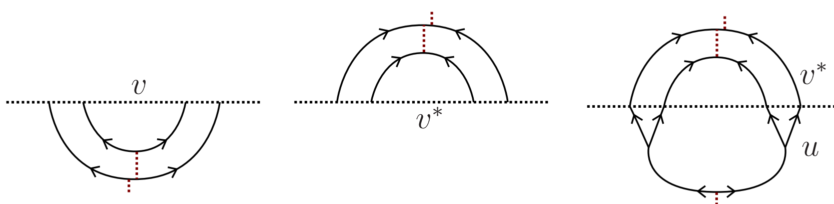


In this way, the boundary of a $\mathfrak{sl}_n$-web can be identified with a $\vec{k}$ as above. The $\mathfrak{sl}_n$-webs without boundary (that is $k_i \in \{0, n\}$) are called *closed* $\mathfrak{sl}_n$-webs.

Important convention: We tend to think in pictures and, by abuse of notation, sometimes call *only* the $\bar{\mathbb{Q}}(q)$-linear generators of $\mathbf{Sp}^n_f(\mathbf{U}_q(\mathfrak{sl}_n))$ (i.e. no formal $\bar{\mathbb{Q}}(q)$-sums, but all possible pictures) $\mathfrak{sl}_n$-webs. Of course, by linearity, these suffice for our purposes.

Moreover, following Brundan and Stroppel [9], we will write v^* to denote the $\mathfrak{sl}_n$-web obtained by reflecting a given $\mathfrak{sl}_n$-web v horizontally and reversing all orientations but keeping the labels fixed. By v^*u (recall that our reading convention is from bottom to top and right to left: First u, then v^*) we shall mean the closed $\mathfrak{sl}_n$-web obtained by gluing v^* on top of u, when such a construction is possible. That is, when the number of strands, the labels and the orientation match at the cut-line. Note that we do not picture the labels below.

(3.2.12)



It is worth noting that, using an analogon of 3.2.11, this way we match Brundan and Stroppel's notation for their multiplication in the generalized arc algebra with the one we mostly use later, since roughly $\mathrm{HOM}_{\mathrm{nh}}(\widehat{u}, \widehat{v}) \cong \mathrm{HOM}_{\mathrm{nh}}(\mathbf{1}, \widehat{v^*u})$ (see also Definition 3.33).

These notions allow us to define a q-sesquilinear form on the $\mathfrak{sl}_n$-web space which we call the *Kuperberg form* or *Kuperberg bracket* of $W_n(\Lambda)$.

Definition 3.22. (**Kuperberg form**) Given $u, v \in W_n(\Lambda)$ we define the *Kuperberg form*

$$\langle \cdot, \cdot \rangle_{\mathrm{Kup}} \colon W_n(\Lambda) \times W_n(\Lambda) \to \bar{\mathbb{Q}}(q)$$

by

$$\langle u, v \rangle_{\mathrm{Kup}} = q^{d(\vec{k})} \mathrm{ev}(v^*u),$$

where the evaluation map $\mathrm{ev}(\cdot) \colon \mathrm{End}_{\dot{\mathbf{U}}_q(\mathfrak{sl}_m)}(n^\ell) \to \bar{\mathbb{Q}}(q)$ is the one obtained by interpreting the closed $\mathfrak{sl}_n$-web v^*u using Theorem 3.17 as an intertwiner with normalization factor $d(\vec{k})$ given by

$$d(\vec{k}) = \frac{1}{2}\left(n(n-1)\ell - \sum_{i=1}^{m} k_i(k_i - 1)\right). \tag{3.2.13}$$

We define the Kuperberg form to be q-antilinear in the first and q-linear in the second entry.

Remark 3.23. Any closed $\mathfrak{sl}_n$-web $w = v^*u$ is because of Theorem 3.17 an intertwiner from the trivial representation to itself, i.e. just a multiplication with a quantum number. But, in contrast to the cases $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$, it is not clear how to compute this number directly. The reason is mostly due to relation 3.2.7.

To be more precise, in the cases $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ one can use these relations to evaluate each closed $\mathfrak{sl}_n$-web w pictorial by collapsing faces step-by-step, since every relation lowers the number of vertices of the $\mathfrak{sl}_n$-web w. This is no longer true for $n > 3$ because of the square switch relation. Hence, for a huge $\mathfrak{sl}_n$-web it is not clear how to perform a sequence of face reducing moves, i.e. 3.2.4 and 3.2.5, to obtain this quantum number. We will give an alternative way to do it using an *algorithm* later in Definition 4.13.

Mackaay and Yonezawa showed in [55] Corollary 4.10 the following. Note that we do not need the q-Shapovalov form very explicitly in this paper and we therefore only refer to e.g. the part before Corollary 4.10 in [55] for the definition.

Proposition 3.24. *The Kuperberg form on W_Λ is, under q-skew Howe duality from Proposition 3.19, exactly the q-Shapovalov form $\langle\cdot,\cdot\rangle_{\mathrm{Shap}}$.* □

3.2.6. *Flow lines.* We will now define $\mathfrak{sl}_n$-flows in the spirit of $\mathfrak{sl}_3$-flows defined by Khovanov and Kuperberg in [37]. We will show that they encode in a combinatorial way the coefficients $c(\vec{k},\vec{S})$ of a $\mathfrak{sl}_n$-web $u \in W_n(\vec{k})$ if one re-writes u explicitly as a sum of elementary tensors $x_{\vec{S}}$ using the identification

$$W_n(\vec{k}) \cong \mathrm{Inv}_{\dot{\mathbf{U}}_q(\mathfrak{sl}_n)}(\Lambda^{\vec{k}}\bar{\mathbb{Q}}^n) \subset \Lambda^{\vec{k}}\bar{\mathbb{Q}}^n = \Lambda^{k_1}\bar{\mathbb{Q}}^n \otimes \cdots \otimes \Lambda^{k_m}\bar{\mathbb{Q}}^n.$$

Later in this paper we give an alternative description using standard n-multitableaux as the author did in the $\mathfrak{sl}_3$ case in [75]. This description turns out to be quite powerful. In fact, a look at Example 3.26 indicates that it is already tricky to find a particular flow line. To find *all* flow lines is a non-trivial task and we use the combinatorial identification of Section 4.1 from n-multitableaux to flow lines to do it.

Note that the translation is exactly saying that the n-multitableaux and their degree's are under q-skew Howe duality nothing else than the action of the F_i's of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ on its weight spaces. But since we need the multitableaux framework in Section 5.1 to connect Hu and Mathas basis to the $\mathfrak{sl}_n$-web algebras (and it is easier to work with them than with the action), we discuss it in detail later. The more algebraic motivated reader is encouraged to work out the corresponding action.

The reader familiar with the notation from Khovanov and Kuperberg [37] or [51] and [75] should be careful that our $\vec{S}$ denotes the $\mathfrak{sl}_n$-*state string* in contrast to the notation J that is used in the papers mentioned before for the $\mathfrak{sl}_3$-state string. Moreover, given a $\mathfrak{sl}_n$-web u, we denote its vertex and edge sets by $V(u)$ and $E(u)$.

Definition 3.25. (**$\mathfrak{sl}_n$-flow lines**) Let $u \in W_n(\vec{k})$ be a $\mathfrak{sl}_n$-web. The set of possible *edge colors* is

$$\mathcal{S} = \mathfrak{P}(\{n,\dots,1\}) = \mathfrak{P}^0(\{n,\dots,1\}) \cup \cdots \cup \mathfrak{P}^n(\{n,\dots,1\}),$$

that is we identify the allowed edge colors with the subsets of $\{n,\dots,1\}$ where we order these colors by the number of their elements. We write $S_j \in \mathcal{S}$ with $S_j = \{s_1,\dots,s_j\}$ if S_j has j elements and its elements are ordered decreasing.

An $\mathfrak{sl}_n$-*flow line f for u* is a coloring of the edges of u such that the following is satisfied.

- If the edge $e \in E(u)$ of u has a label j, then the color has to be a j-element subset.

- Recall that at each vertex there are either two incoming or outgoing edges. The colors for these two edges S, S' have to be disjoint, i.e. $S \cap S' = \emptyset$.
- The unique outgoing or incoming edge S'' has to satisfy $S'' = S \cup S'$.

For each vertex v define the *weight* $\mathrm{wt}^v(u_f)$ to be $\ell(S,S') = |\{(i,j) \mid i \in S, j \in S', i < j\}|$ iff S, S' are the two upper edges and $-\ell(S', S)$ iff S, S' are the two lower edges (in both cases ordered from left to right). Here, and in the following, u_f denotes a $\mathfrak{sl}_n$-web u *together* with a fixed flow f for the $\mathfrak{sl}_n$-web u.

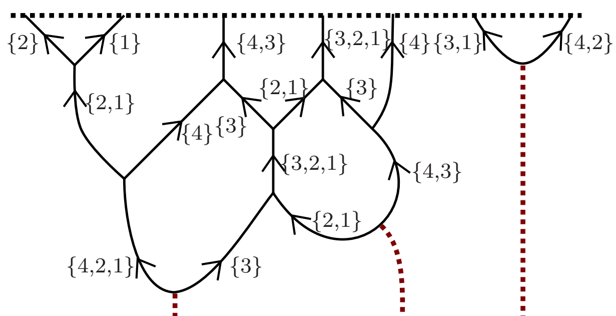
In the dual cases, that is with all arrows reversed, we flip all the sign conventions from above.

The (total) weight $\mathrm{wt}(u_f)$ is defined to be the sum over all local weights, i.e.

$$\mathrm{wt}(u_f) = \sum_{v \in V(u)} \mathrm{wt}^v(u_f).$$

The *state string* $\vec{S}_{u_f}$ given by u_f is defined to be the ordered tupel of the colors of u_f that touch the cut-line. Note that $\vec{S}_{u_f}$ corresponds 1 to 1 to a n-multipartition. This identification is non-trivial and part of Section 4.1, i.e. we identify flows on $\mathfrak{sl}_n$-webs u_f and n-multitableaux $\iota(u_f) \in \mathrm{Std}(\vec{\lambda})$ and the corresponding n-multipartition λ belongs to the flow on the boundary $\vec{S}_{u_f}$.

Example 3.26. For example, if $n = 4$, $\vec{k} = (1,1,0,2,3,1,2,2)$ and the $\mathfrak{sl}_n$-web u is the one from above, then a $\mathfrak{sl}_n$-flow line for u is for example



Moreover, the weight in this case (as the reader is encouraged to check) is, if we read from top to bottom and left to right, the local sum of the weights

$$\mathrm{wt}(u_f) = 0 - 1 + 0 + 3 + 0 + 1 + 2 - 2 + 4 + 2 = 9.$$

Remark 3.27. In the cases $n = 2, 3$ one can think of the subsets of $\{n, \ldots, 1\}$ as honest colors: For edges with label 0 or n one has no choice, since the only possible subsets of size 0 or n are $\emptyset$ and $\{n, \ldots, 1\}$ respectively. So a reasonable convention in those cases is not to picture the flow at all.

For $n = 2$ this convention reduces the number of needed colors from $2^2 = 4$ to 2, i.e. one only needs to specify if one uses $\{1\}$ or $\{2\}$ for edges labeled 1. These correspond exactly to the orientations used by Brundan and Stroppel in their sequence of papers [9], [10], [11], [12] and [13] by saying that a counter-clockwise orientation corresponds to the subset $\{1\}$.

For $n = 3$ this reduces the number of colors to 6 and those are exactly the flow lines introduced by Khovanov and Kuperberg [37] and used for example in [51] by saying that $\{1\}$ and $\{2,1\}$ are pictured as flows in the same direction as the orientation of the corresponding edge, $\{3\}$ and $\{3,2\}$ to a flow in the opposite direction and $\{2\}$ and $\{3,1\}$ to no flow at all.

Using these conventions we obtain from Theorem 3.17 an $\mathfrak{sl}_n$ analogue of Khovanov and Kuperberg's results in the $\mathfrak{sl}_3$ case. By abuse of notation, we consider the $\mathfrak{sl}_n$-web $u \in W_n(\vec{k})$ below directly as an element of $\mathrm{Inv}_{\dot{\mathbf{U}}_q(\mathfrak{sl}_n)}(\Lambda^{\vec{k}}\bar{\mathbb{Q}}^n)$.

Theorem 3.28. *Let $\vec{k} \in \Lambda(m, n\ell)_n$ for some $\ell \in \mathbb{N}$. Fix a $\mathfrak{sl}_n$-web $u \in W_n(\vec{k})$. Let us denote by $Fl(u)$ the set of all possible flow lines of u. Then*

$$u = \sum_{u_f \in Fl(u)} (-q)^{\mathrm{wt}(u_f)} x_{\vec{S}_{u_f}} \quad \text{with } x_{\vec{S}_{u_f}} \in \Lambda^{\vec{k}}\bar{\mathbb{Q}}^n, \tag{3.2.14}$$

where the pair $(\vec{S}_{u_f}, \mathrm{wt}(u_f))$ is the state string and weight of u_f and $x_{\vec{S}_{u_f}}$ is the corresponding elementary tensor.

Proof. This is just the assembling of pieces now. To be more precise, we can use induction on the number of vertices of u where it is easy to check for all small cases $V(u) < 2$.

The main observation now is that locally our conventions match with the ones given above Definition 3.16 for the intertwiners $M_s^{a,b}$ and $M_m^{a,b}$ and their duals (recall that we had flipped the sign convention in those cases). It is worth noting that the exponents for $M_s^{a,b}$ equals exactly our definition, since for $T \subset S$ we see that $\ell(S, T) = \ell(S - T, T)$ and that our convention how flow lines "add" around vertices also matches exactly with the cases where the intertwiner map to a non-trivial element. Thus, summing over all possibilities is the same as taking all possible flows.

We can proceed by induction from a smaller $\mathfrak{sl}_n$-web to a bigger $\mathfrak{sl}_n$-web by adding one vertex. This is the exactly the same as composing the intertwiner for the smaller $\mathfrak{sl}_n$-web with one of the maps from above. Note that the coefficients will be multiplied. Hence, their powers add and this happens in exactly the same way as for the total weight. $\square$

Example 3.29. In the case of the flow given in Example 3.26 we see that the weight is 9 and the state string is $\vec{S}_{u_f} = (\{2\}, \{1\}, \emptyset, \{4, 3\}, \{3, 2, 1\}, \{4\}, \{3, 1\}, \{4, 2\})$. Hence, the corresponding elementary tensor is

$$x_{\vec{S}_{u_f}} = x_2 \otimes x_1 \otimes 1 \otimes (x_4 \wedge x_3) \otimes (x_3 \wedge x_2 \wedge x_1) \otimes x_4 \otimes (x_3 \wedge x_1) \otimes (x_4 \wedge x_2).$$

It is an element of $\Lambda^{\vec{k}}\bar{\mathbb{Q}}^4 = \bar{\mathbb{Q}}^4 \otimes \bar{\mathbb{Q}}^4 \otimes \bar{\mathbb{Q}} \otimes \Lambda^2\bar{\mathbb{Q}}^4 \otimes \Lambda^3\bar{\mathbb{Q}}^4 \otimes \bar{\mathbb{Q}}^4 \otimes \Lambda^2\bar{\mathbb{Q}}^4 \otimes \Lambda^2\bar{\mathbb{Q}}^4$, since we have $\vec{k} = (1, 1, 0, 2, 3, 1, 2, 2)$. The Theorem 3.28 ensures that it appears in the decomposition of u as a sum of elementary tensors at least once with multiplicity $(-q)^{\mathrm{wt}(u_f)} = -q^9$. In order to find the full coefficient for $x_{\vec{S}_{u_f}}$ one has to know all flows with the same state string as u_f and their weights.

3.3. KL-R algebras, categorification of $\mathfrak{sl}_n$-webs and categorified q-skew Howe duality.

3.3.1. *The special quantum 2-algebras.* Khovanov-Lauda and independently Rouquier introduced certain diagrammatic 2-categories $\mathcal{U}(\mathfrak{g})$ which categorify the integral version of the corresponding idempotented quantum groups, see [40] or [65].

In addition, Cautis and Lauda [21] defined diagrammatic 2-categories $\mathcal{U}_Q(\mathfrak{g})$ with certain scalars Q consisting of t_{ij}, r_i and s_{ij}^{pq} which determine possible different choices in the "KL-R part" of the categorified quantum groups.

We briefly recall $\mathcal{U}(\mathfrak{sl}_m) = \mathcal{U}_Q(\mathfrak{sl}_m)$ in this section. Much more can be found in the papers cited above. We fix the following possible choices: The scalars Q are given by $t_{ij} = -1$ if $j = i + 1$, $t_{ij} = 1$ otherwise, $r_i = 1$ and $s_{ij}^{pq} = 0$ (this choice corresponds exactly to the one from [55] and also corresponds to the signed version in [40] and [41]).

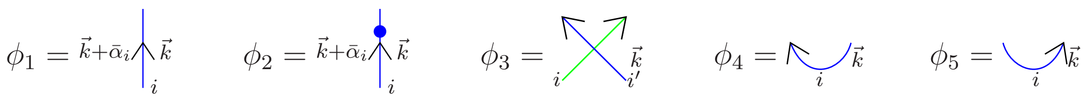
It is worth noting that we again try to distinguish between weights $\vec{k}$ and partitions λ. Moreover, we again restrict ourself for simplicity to $\bar{\mathbb{Q}}$ as the underlying field. Recall that $\bar{\alpha}$ was given by applying Equation 3.2.8 to the simple $\mathfrak{gl}_m$-roots α_i used before.

Definition 3.30. (**Khovanov-Lauda**) The 2-category $\mathcal{U}(\mathfrak{sl}_m)$ is defined as follows.

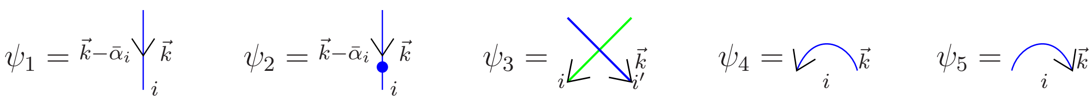
- The objects in $\mathcal{U}(\mathfrak{sl}_m)$ are the weights $\vec{k} \in \mathbb{Z}^{m-1}$.

For any pair of objects $\vec{k}$ and $\vec{k}'$ in $\mathcal{U}(\mathfrak{sl}_m)$, the hom category $\mathcal{U}(\mathfrak{sl}_m)(\vec{k}, \vec{k}')$ is the $\mathbb{Z}$-graded, additive $\bar{\mathbb{Q}}$-linear category consisting of the following data.

- Objects (or 1-morphisms), that is finite formal sums of the form $\mathcal{E}_{\underline{i}}\mathbf{1}_{\vec{k}}\{t\}$ and $\mathcal{F}_{\underline{i}}\mathbf{1}_{\vec{k}}\{t\}$ where $t \in \mathbb{Z}$ is a grading shift and $\underline{i}$ is string of $i \in \{1, \ldots, n-1\}$ such that $\vec{k}' = \vec{k} + \sum_{a=1}^{l} \epsilon_a i_a'$.
- The spaces of morphisms (or of 2-morphisms) are graded, $\bar{\mathbb{Q}}$-vector spaces generated by compositions of diagrams shown below. Here $\{t\}$ denotes a degree shift up by t and we use the shorthand notations $\alpha^{ii'} = (\bar{\alpha}_i, \bar{\alpha}_{i'})$ and $\alpha^{\vec{k}i} = 2\frac{(\vec{k}, \bar{\alpha}_i)}{(\bar{\alpha}_i, \bar{\alpha}_i)}$.



with $\phi_1 = \mathrm{id}_{\mathcal{E}_i\mathbf{1}_{\vec{k}}}$, $\phi_2 \colon \mathcal{E}_i\mathbf{1}_{\vec{k}} \Rightarrow \mathcal{E}_i\mathbf{1}_{\vec{k}}\{\alpha^{ii}\}$, $\phi_3 \colon \mathcal{E}_i\mathcal{E}_{i'}\mathbf{1}_{\vec{k}} \Rightarrow \mathcal{E}_{i'}\mathcal{E}_i\mathbf{1}_{\vec{k}}\{\alpha^{ii'}\}$ and cups and caps $\phi_4 \colon \mathbf{1}_{\vec{k}}\{\frac{1}{2}\alpha^{ii} + \alpha^{\vec{k}i}\} \Rightarrow \mathcal{E}_i\mathcal{F}_i\mathbf{1}_{\vec{k}}$ and $\phi_5 \colon \mathbf{1}_{\vec{k}}\{\frac{1}{2}\alpha^{ii} - \alpha^{\vec{k}i}\} \Rightarrow \mathcal{F}_i\mathcal{E}_i\mathbf{1}_{\vec{k}}$. Moreover, we have diagrams of the form



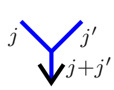
with $\psi_1 = \mathrm{id}_{\mathcal{F}_i\mathbf{1}_{\vec{k}}}$, $\psi_2 \colon \mathcal{F}_i\mathbf{1}_{\vec{k}} \Rightarrow \mathcal{F}_i\mathbf{1}_{\vec{k}}\{\alpha^{ii}\}$, $\psi_3 \colon \mathcal{F}_i\mathcal{F}_{i'}\mathbf{1}_{\vec{k}} \Rightarrow \mathcal{F}_{i'}\mathcal{F}_i\mathbf{1}_{\vec{k}}\{\alpha^{ii'}\}$ and cups and caps $\psi_4 \colon \mathcal{F}_i\mathcal{E}_i\mathbf{1}_{\vec{k}} \Rightarrow \mathbf{1}_{\vec{k}}\{\frac{1}{2}\alpha^{ii} + \alpha^{\vec{k}i}\}$ and $\psi_5 \colon \mathcal{E}_i\mathcal{F}_i\mathbf{1}_{\vec{k}} \Rightarrow \mathbf{1}_{\vec{k}}\{\frac{1}{2}\alpha^{ii} - \alpha^{\vec{k}i}\}$.

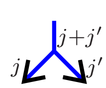
The convention for reading these diagrams is from right to left and bottom to top. The relations on the 2-morphisms are those of the signed version in [40] and [41], i.e. the 2-morphisms should satisfy several relation which we will not recall here since we do not need them explicitly. Details (for all possible Q) can be for example found in Cautis and Lauda's paper [21].

Recall that, given a 1-category $\mathcal{C}$, then the objects of the *Karoubi envelope* ***Kar***$(\mathcal{C})$ are pairs (O, e) where $O \in \mathrm{Ob}(\mathcal{C})$ is an object and $e \colon O \to O$ is a projector $e^2 = e$. For the case we are interested in, that is the Karoubi envelope of $\mathcal{U}(\mathfrak{sl}_2)$ (which is usually denoted $\dot{\mathcal{U}}(\mathfrak{sl}_2)$ and not $\mathbf{Kar}(\mathcal{U}(\mathfrak{sl}_2))$), one can define analoga of the *divided powers* $E_i^{(j)}$ and $F_i^{(j)}$ as follows (these are 1-morphisms, aka objects of the hom-spaces).

Fix a "*color*" $j \in \mathbb{N}$ and set $O = \mathcal{F}^j\mathbf{1}_{\vec{k}}$. Define $e_j \colon O \to O$ to be the idempotent obtained by any reduced presentation of the longest braid word on j strands together with a certain, fixed dot placement (see 2.18 in [42]). Then $\mathcal{F}^{(j)}\mathbf{1}_{\vec{k}} = (O\{\frac{j(j-1)}{2}\}, e_j)$ and one can define $\mathcal{E}^{(j)}\mathbf{1}_{\vec{k}}$ similar.

The category $\dot{\mathcal{U}}(\mathfrak{sl}_2)$ can be described by using *thick calculus*. The (for us) most important 2-morphisms are then given by (the right face should carry the label $\vec{k}$)

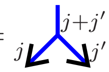
$\colon \mathcal{F}^{(j)}\mathbf{1}_{\vec{k}} \to \mathcal{F}^{(j)}\mathbf{1}_{\vec{k}}$,


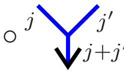
$\colon \mathcal{F}^{(j+j')}\mathbf{1}_{\vec{k}} \to \mathcal{F}^{(j)}\mathcal{F}^{(j')}\mathbf{1}_{\vec{k}}$,


$\colon \mathcal{F}^{(j)}\mathcal{F}^{(j')}\mathbf{1}_{\vec{k}} \to \mathcal{F}^{(j+j')}\mathbf{1}_{\vec{k}}$

called *thick identity*, *split* and *merge* (recall that we read from bottom to top). The *thick crossing* is then a composite of (first) the merge and (then) the split

(3.3.1) 

$: \mathcal{F}^{(j)}\mathcal{F}^{(j')}\mathbf{1}_{\vec{k}} \to \mathcal{F}^{(j')}\mathcal{F}^{(j)}\mathbf{1}_{\vec{k}} =$ 

$\circ$ 

The precise definitions are not important for us and can be found in Chapter 2 of [42]. There are similar definitions for the upwards pointing arrows as well. Note that the split and the merge are of degree $-jj'$. The 2-category consisting of these diagrams is denoted by $\check{\mathcal{U}}(\mathfrak{sl}_2)$.

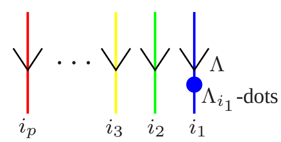
The 2-category $\check{\mathcal{U}}(\mathfrak{sl}_2)$ can then be extended to a graphical calculus for $\dot{\mathcal{U}}(\mathfrak{sl}_2)$ by introducing a generalized version of the dot-2-morphisms: For each each j-labeled thick strand one allows now a symmetric polynomial $p \in \mathbb{Z}[X_1, \dots, X_j]^{S_j}$ which satisfy certain relations, see [42] for details.

But there is *no combinatorial* description for the 2-category $\dot{\mathcal{U}}(\mathfrak{sl}_m)$ yet. So we only define $\check{\mathcal{U}}(\mathfrak{sl}_m)$ as a full 2-subcategory of $\dot{\mathcal{U}}(\mathfrak{sl}_m)$ with the same objects $\vec{k}$, but with 1-morphisms generated by the divided powers $\mathcal{E}_i^{(j)}\mathbf{1}_{\vec{k}}$ and $\mathcal{F}_i^{(j)}\mathbf{1}_{\vec{k}}$ from above for each $i \in \{1, \dots, n-1\}$.

3.3.2. *The cyclotomic KL-R algebras.* We, again very briefly, recall the definition of the so-called *cyclotomic KL-R algebras* of type A, due to Khovanov and Lauda [38], [39] and independently Rouquier [65]. Moreover, we recall very shortly Hu and Mathas graded cellular basis [30] for these algebras.

Let Λ be a dominant $\mathfrak{sl}_m$-weight, V_Λ the irreducible $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-module of highest weight Λ and P_Λ the set of weights in V_Λ. For us Λ will usually denote ℓ-times the n-th fundamental $\mathfrak{sl}_m$-weight.

Definition 3.31. (**Khovanov-Lauda, Rouquier**) The *cyclotomic KL-R algebra* R_Λ is defined as the 2-subalgebra of $\mathcal{U}(\mathfrak{sl}_m)$ consisting of all diagrams with only downward oriented strands and right-most region labeled Λ modded out by the ideal generated by all diagrams of the form

(3.3.2) 

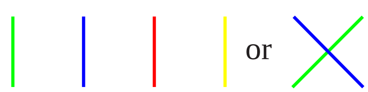
This relation is known as the *cyclotomic relation*.

Note that

$$R_\Lambda = \bigoplus_{\vec{k} \in P_\Lambda} R_\Lambda(\vec{k}),$$

where $R_\Lambda(\vec{k})$ is the subalgebra generated by all diagrams whose left-most region is labeled $\vec{k}$. It is not clear from the definition what the dimension of R_Λ is. Moreover, it is not clear that R_Λ is finite dimensional, but Brundan and Kleshchev proved that R_Λ is indeed finite dimensional [6].

It is worth noting that, if we draw pictures for the cyclotomic KL-R algebra, then we do not need orientations anymore, that is pictures will look like



In [30] Hu and Mathas gave a graded cellular basis of the cyclotomic KL-R algebra R_Λ. We do not recall their definition here, since it is not short and we give an alternative definition in our language later. The reader is encouraged to take a look in their great paper. We call their

basis *HM-basis*. We only mention that their basis (in the form we need it) is parametrized by $\vec{\lambda} \in \Lambda^+(c, c(\vec{k}), c')$, i.e. all c'-multipartitions of $c(\vec{k})$ for all suitable c, c', and $\vec{T}, \vec{T}' \in \mathrm{Std}(\vec{\lambda})$, i.e. standard c'-multitableaux. They denote their basis by

$$\{\psi^{\vec{\lambda}}_{\vec{T}',\vec{T}} \mid \vec{\lambda} \in \mathfrak{P}_{c(\vec{k})} \text{ and } \vec{T}, \vec{T}' \in \mathrm{Std}(\vec{\lambda})\}, \tag{3.3.3}$$

where $\mathfrak{P}_{c(\vec{k})}$ is the set of all multipartitions of $c(\vec{k})$. Moreover, the basis is graded by

$$\deg_{\mathrm{BKW}}(\psi^{\vec{\lambda}}_{\vec{T}',\vec{T}}) = \deg_{\mathrm{BKW}}(\vec{T}) + \deg_{\mathrm{BKW}}(\vec{T}'),$$

where the degree is Brundan, Kleshchev and Wang's degree given in [8], which we recall in 3.5. It is worth noting that we sometimes like to think of the cyclotomic KL-R algebra as the (graded) cyclotomic Hecke algebra using Brundan and Kleshchev's graded isomorphism [6].

To make the connection with our context: For us we fix $c' = n$ in the context of $\mathfrak{sl}_n$-webs. And we should mention that $c(\vec{k})$ is a constant that only depends on the weight $\vec{k}$. It could be written in an explicit formula as the author has done in [75] for $\mathfrak{sl}_3$, but we do not do it here since we do not use the formula and it is rather cumbersome. We only note that it just counts the number of F's one has to apply (as an $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-action) to go from (n^ℓ) to $\vec{k}$.

And the constant $c = c(\vec{S})$ depends only on the $\mathfrak{sl}_n$-flows at the cut-line (and can be also written down explicitly, but we do not do it). To summarize, we have two fixed numbers n and $c(\vec{k})$ and consider the set of all n-multipartitions of $c(\vec{k})$.

Definition 3.32. (**Thick cyclotomic KL-R**) The *thick cyclotomic KL-R algebra*, denoted by $\check{R}_\Lambda$, is the 2-subquotient of $\check{\mathcal{U}}(\mathfrak{sl}_m)$ defined by the 2-subalgebra of all diagrams with only downward oriented strands and right-most region labeled Λ and modded out by the cyclotomic relation 3.3.2. We note that, since a strand of thickness j can also be seen as a certain idempotent on j-vertical strands (see Section 2 in [42]), this induces relations on "thick dots" as well.

We should note that it is not clear from the definition above that all the relations in the $\mathfrak{sl}_n$-foam or matrix factorization set-up follow from the thick cyclotomic KL-R algebra. We show this non-trivial fact in an indirect way in Theorem 3.36 by showing that our $\mathfrak{sl}_n$-web version of HM-basis that we give in Definition 5.10 by an growth algorithm comes from a thick version of the HM-basis for the thick KL-R algebra $\check{R}_\Lambda$.

For $n = 2$ (if we go to the Bar-Natan cobordism setting) this implies under q-skew Howe the "facet with two dots equals zero" relation given in [2] and for $n = 3$ the "facet with three dots equals zero".

It is worthwhile to note that the cyclotomic relation implies later the *finite dimensionality* of the $\mathfrak{sl}_n$-link homologies. Roughly: The cyclotomic KL-R *suffices* for the $\mathfrak{sl}_n$-link homologies.

3.3.3. *Matrix factorizations and categorification of $\mathfrak{sl}_n$-webs.* We very briefly recall the notion of *matrix factorizations* in this section. Furthermore, we also very briefly recall how they *categorify* the $\mathfrak{sl}_n$-webs. In fact, we will only explain where the reader can find the algebraic definition for our notation. The reason for this is that recalling all the details will highly increase the number of pages of our paper (which is already too long anyway) and one of our main points is that we do not want to use the notion of matrix factorizations, but the q-skew Howe dual instead. One reason why this is possible is in fact the well-definedness of the 2-functor in Theorem 3.35.

Our main source is the paper of Mackaay and Yonezawa [55] and the paper of Mackaay [50] where the reader can find much more details. We keep our notation close to theirs (e.g. we suppress the shifts in homology degree) and the corresponding algebraic definitions can be found there.

It is worth noting that matrix factorizations in the context of $\mathfrak{sl}_n$-webs and link invariants were introduced by Khovanov and Rozansky in [43]. Later their constructions were independently generalized by Wu in [79] and Yonezawa in [80] and [81].

We note that everything can be done more topological and combinatorial using the $\mathfrak{sl}_n$ analogon of Bar-Natan's $\mathfrak{sl}_2$-cobordisms and Khovanov's $\mathfrak{sl}_3$-foams. We use the algebraic notion of matrix factorizations here, because in the writing process of this paper it was not clear what a complete list of relations in the category of these $\mathfrak{sl}_n$-*foams* is. This, is settled now, see Queffelec and Rose [61]. Now it is no big problem anymore to follow our approach here with their "$\mathfrak{sl}_n$-foamation".

All the reader needs to know on the level of $\mathfrak{sl}_n$-webs is that a $\mathfrak{sl}_n$-web u *without* tags can be seen as a matrix factorization denoted by $\widehat{u}$. Such matrix factorizations are $(\mathbb{Z}/2\mathbb{Z},\mathbb{Z})$-graded where the latter one is called the q-*grading*. Shifting in the first grading is indicated by $\langle\cdot\rangle$ and shift the q-grading by $\{\cdot\}$. For example there is the notion of the dual matrix factorization $\widehat{u}_{\bullet}$ and one can check that $\widehat{u}_{\bullet}\cong\widehat{u^*}\langle 1\rangle\{d(\vec{k})\}$ for $u\in W_n(\vec{k})$.

Very important for us in the following are the ones that correspond to an $E_i^{(j)}$ or to an $F_i^{(j)}$. Both of them are indecomposable. We denote them by $\widehat{E}^{(j)}_{(k_i,k_{i+1})}$ and $\widehat{F}^{(j)}_{(k_i,k_{i+1})}$ respectively (note that Mackaay and Yonezawa [55] and Mackaay [50] use E_- instead of F). Furthermore, we denote the one that corresponds to the identity by $\widehat{1}_{\vec{k}}$.

We *freely* switch between the notions of $\mathfrak{sl}_n$-webs and their corresponding matrix factorizations (e.g. we tend to write $F_i^{(j)}$ instead of $\widehat{F}^{(j)}_{(k_i,k_{i+1})}$). The reason is that first ones are combinatorial and easier to work with. In short, on the level of 1-morphism we usually use the language of $\mathfrak{sl}_n$-webs, but on the level of 2-morphism we use the language explained below, i.e. using certain EXT-spaces which are isomorphic to certain $\langle\cdot\rangle$-shifted HOM-spaces (modulo null-homotopic maps) between matrix factorizations (see Proposition 5.6 in [55]). Thus, we can *loosely* call them *homomorphisms of matrix factorizations*.

3.3.4. *The $\mathfrak{sl}_n$-web-algebra.* Now we recall the definition of the $\mathfrak{sl}_n$-web algebra $H_n(\vec{k})$ from [50].

Definition 3.33. (**Mackaay: The $\mathfrak{sl}_n$-web algebra**) Choose a fixed monomial basis $B(W_n(\vec{k}))$ of $W_n(\vec{k})$. That is, any basis vector $u\in B(W_n(\vec{k}))$ can be obtained from a fixed highest weight vector using q-skew Howe duality. We do not recall the exact definition here and refer to Example 4.1 instead. It should be noted that this includes that any basis vector is one fixed $\mathfrak{sl}_n$-web without any quantum factors.

For any pair $u,v\in B(W_n(\vec{k}))$, define (for $d(\vec{k})$ as in 3.2.13)

$$ {}_vH_n(\vec{k})_u=\mathrm{EXT}(\widehat{u},\widehat{v})\cong H(\widehat{v^*u})\{d(\vec{k})\}. $$

The $\mathfrak{sl}_n$-web algebras $H_n(\vec{k})$ and $H_n(\Lambda)$ are defined by

$$ H_n(\vec{k})=\bigoplus_{u,v\in B(W_n(\vec{k}))}{}_vH_n(\vec{k})_u\quad\text{and}\quad H_n(\Lambda)=\bigoplus_{\vec{k}\in\Lambda(m,n\ell)_n}H_n(\vec{k}), $$

with multiplication induced by the composition of maps between the corresponding matrix factorizations. It should be noted that $H_n(\vec{k})$ is a $\mathbb{Z}$-graded, finite dimensional, associative algebra with

unit. Moreover, the algebra is a $\mathbb{Z}$-graded, symmetric Frobenius algebra of Gorenstein parameter $2d(\vec{k})$, that is, $H_n(\vec{k})\{-2d(\vec{k})\}$ is graded isomorphic (as $H_n(\vec{k})$-bimodules) to its graded dual. The trace τ is given by pairing elements of $H_n(\vec{k})$ with the identity $1 = \sum_{u \in W_n(\vec{k})} \mathrm{id}(\widehat{u})$.

Remark 3.34. In [50] Mackaay has chosen a certain monomial basis called *LT-basis*. This basis is obtained from a q-skew Howe analogon of an *intermediate crystal basis* defined by Leclerc and Toffin [46]. We note that all of Mackaay's constructions that are important for us only depend on the fact that this basis is monomial. In fact, Mackaay's arguments in Lemma 7.5 in [50] show that, for all choices of bases, all the possibly different $\mathfrak{sl}_n$-web algebras will be Morita equivalent.

3.3.5. *Categorified q-skew Howe duality.* As a last ingredient we are going to recall now how these construction can be used to categorify an instance of q-skew Howe duality. We should note that this is in fact one of our main ingredients, but since the definition of the 2-action of $\mathcal{U}(\mathfrak{sl}_m)$ on $\dot{\mathcal{W}}^{\circ}_{\Lambda} \cong \mathcal{W}^{p}_{\Lambda}$ (the first is a 2-category of matrix factorizations and the second is a 2-category of $H_n(\Lambda)$-representations, see [50] Definition 7.1) is not short in any sense, we only recall it very briefly, i.e. by an example of the action on 2-morphisms. The full list can be found in Section 9 of Mackaay and Yonezawa's paper [55].

The point is that categorified q-skew Howe duality also defines an 2-action of $\mathcal{U}(\mathfrak{sl}_m)$ on $\mathcal{W}^{p}_{\Lambda}$.

Theorem 3.35. ***(Categorified pictorial** q**-skew Howe duality** - Theorem 9.7 in* [55]*) The 2-functor*

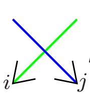
$$\Gamma_{m,n\ell,n} \colon \mathcal{U}(\mathfrak{sl}_m) \to \mathcal{W}^{p}_{\Lambda}, \tag{3.3.4}$$

which is defined on objects and 1-morphisms the same way as the one from Proposition 3.19 and on 2-morphisms by the list of cases given in Section 9 in [55]*, is a well-defined 2-action of $\mathcal{U}(\mathfrak{sl}_m)$ on $\mathcal{W}^{p}_{\Lambda}$ giving latter the structure of a strong $\mathfrak{sl}_m$-2-representation in the sense of* [21]*. This strong $\mathfrak{sl}_m$-2-representation induces an additive equivalence of 2-categories*

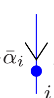
$$\tilde{\Gamma}_{m,n\ell,n} = \tilde{\Gamma} \colon \ \mathrm{R}_{\Lambda}\text{-}\mathrm{p}\boldsymbol{Mod}_{\mathrm{gr}} \to \mathcal{W}^{p}_{\Lambda}, \tag{3.3.5}$$

i.e. from the category of finite dimensional, $\mathbb{Z}$-graded, projective R_{Λ}-modules to $\mathcal{W}^{p}_{\Lambda}$.

All the reader needs to know to understand the reasoning in this paper about the list for the 2-action is that there are certain homomorphisms between matrix factorizations associated to the for us most important pieces



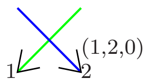
$$i \quad j \quad \vec{k} \mapsto \begin{cases} \widehat{CR}_{ji} \colon \widehat{F}_i\widehat{F}_{i\pm 1} \to \widehat{F}_{i\pm 1}\widehat{F}_i, & \text{if } j = i \pm 1, \\ \widehat{I}_{ii}\widehat{D}_{ii} \colon \widehat{F}_i\widehat{F}_i \to \widehat{F}_i\widehat{F}_i, & \text{if } i = j, \\ \widehat{s}_{ji} \colon \widehat{F}_i\widehat{F}_j \to \widehat{F}_j\widehat{F}_i, & \text{if } |i-j| > 1, \end{cases} \quad \text{and} \quad \vec{k}-\bar{\alpha}_i \ \vec{k} \ i \mapsto \widehat{t}_i \colon \widehat{F}_i \to \widehat{F}_i,$$



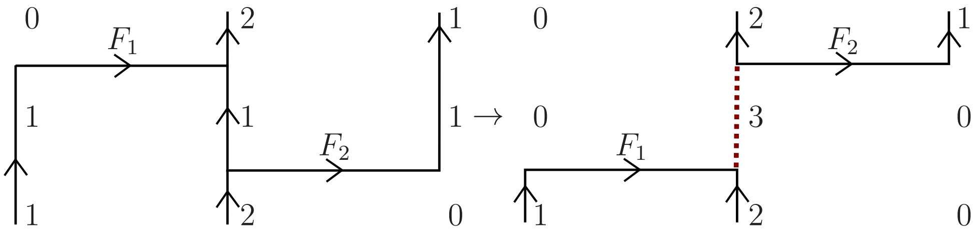
of q-degree 1, -2, 0 and 2 respectively. For the case $n = 2$ these correspond in the familiar *cobordism language* (see for example [45]) to a *saddle*, a *cup* followed by a *cap* and a *shift*. In the $n = 3$ case these can also be translated to natural pictures (see for example [45] or [51]). Moreover, the homomorphism $\widehat{t}_i$ is of q-degree 2 and can be though of *"placing a dot"* on the corresponding ladder. To make the notation cumbersome we use sub- and superscripts like $\widehat{F}^{(j)}_{p,i,\vec{k}}$ to indicate the position p (read from right to left in the KL-R picture and from bottom to top in the $\mathfrak{sl}_n$-web picture), the (possible divided) power j, the residue (or color) i and the weight $\vec{k}$. We sometimes skip some of them and hope that it is clear from the context in those cases.

The 2-action works roughly as we try to illustrate now. Given one of the 2-cell generators of $\mathcal{U}(\mathfrak{sl}_m)$, one has an object given by the $\vec{k}$ and two $\mathfrak{sl}_n$-webs at the bottom u_b and top u_t by reading

from right to left and apply an E_i for each upwards pointing string with label i one passes and an F_i for each downwards pointing string with label i. Then assign a certain homomorphism between the matrix factorization $\widehat{u}_b$ and $\widehat{u}_t$ as a 2-morphism. For example for $n = 3$ and position $p = 1$

$\psi_3 =$  $\mapsto \widehat{CR}_{1,21} \colon u_b = F_1 F_2 v_{(1,2,0)} \to F_2 F_1 v_{(1,2,0)} = u_t$

where the $\widehat{CR}_{1,21}$ is a certain homomorphism between the matrix factorizations. In pictures

$\widehat{CR}_{1,21} \colon$ 

For the reader familiar with the corresponding foamation (see [45], [51] or [61]) we note that this is like "zipping" certain edges away.

Theorem 3.36. *The* 2*-functor* $\Gamma_{m,n\ell,n}$ *extends to a* 2*-functor*

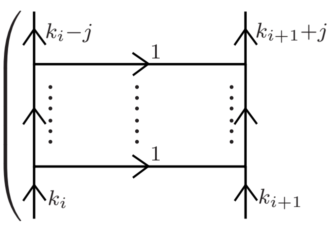
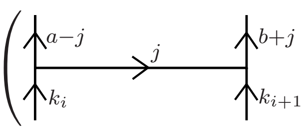
$$\check{\Gamma}_{m,n\ell,n} \colon \check{\mathcal{U}}(\mathfrak{sl}_m) \to \mathcal{W}^p_\Lambda.$$

Proof. Given any two 1-categories and a 1-functor $\mathcal{FUN} \colon \mathcal{C} \to \mathcal{D}$, there exists (by the universal property of the Karoubi envelope) an extension $\overline{\mathcal{FUN}} \colon \mathbf{Kar}(\mathcal{C}) \to \mathbf{Kar}(\mathcal{D})$. Moreover, any category $\mathcal{C}$ embeds via $O \mapsto (O, \mathrm{id})$ fully faithful into $\mathbf{Kar}(\mathcal{C})$. Both statements are still true in the 2-categorical setting.

Thus, it suffices to show that (and the same for $\mathcal{E}_i^{(j)} \mathbf{1}_{\vec{k}}$)

$$\overline{\Gamma}_{m,n\ell,n}(\mathcal{F}_i^{(j)} \mathbf{1}_{\vec{k}}) \cong (\widehat{F}^{(j)}_{(k_i,k_{i+1})}, \mathrm{id}(\widehat{F}^{(j)}_{(k_i,k_{i+1})})), \text{ with } \vec{k} = (\dots, k_i, k_{i+1}, \dots).$$

On the level of the $\mathfrak{sl}_n$-webs this means we need to prove

$\left(\right.$  $, \widehat{I}_i^{(j)} \circ \widehat{D}_i^{(j)} \circ \widehat{t}^{\mathrm{sym}} \left.\right) \cong \left(\right.$  $, \mathrm{id}(\widehat{F}^{(j)}_{(k_i,k_{i+1})}) \left.\right)$

where the ladders labeled 1 are repeated j-times. Here we introduce some notation. We define $\widehat{I}_i^{j'} \colon \widehat{F}_i^{(j'+1)} \to \widehat{F}_i^{(j')} \widehat{F}_i$, $\widehat{I}_i^{(j)} = \widehat{I}_i^1 \circ \dots \circ \widehat{I}_i^{j-1}$, $\widehat{D}_i^{j'} \colon \widehat{F}_i \widehat{F}_i^{(j')} \to \widehat{F}_i^{(j'+1)}$ and $\widehat{D}_i^{(j)} = \widehat{D}_i^{j-1} \circ \dots \circ \widehat{D}_i^1$. The steps $\widehat{I}^{j'}$ and $\widehat{D}^{j'}$ should be composites of $\widehat{CR}$'s and $\widehat{t}$'s exactly as the and $(j',1)$-splitters and $(1,j')$-merges are defined in Section 2 of [42]. The subscript sym should indicate a symmetric spread of dots starting with $j-1$ for the top edge to no dots for the bottom.

Now comes the good part about matrix factorizations: A lot of calculations are already done. So we do not need to re-do them. In fact, the isomorphism above follows from work of Mackaay and Yonezawa [55] (we also mention Wu [79] and Yonezawa [80], [81] here) without any extra calculations. To be precise, Theorem 3.35 implies that $\overline{\Gamma}_{m,n\ell,n}(\mathcal{F}_i^{(j)} \mathbf{1}_{\vec{k}})$ is given as above and Corollary 9.8 in [55] implies that Equation 3.2.6 is satisfied in $K_0^{\oplus}(W^p_\Lambda)$. Thus, there has to be a suitable isomorphism which finishes the proof. $\square$

4. THE UNCATEGORIFIED STORY

4.1. Multitableaux and $\mathfrak{sl}_n$-webs.

4.1.1. *Short overview.* The goal of this section is to describe the connections between $\mathfrak{sl}_n$-webs[8] and n-multitableaux. We focus on the combinatorics at the beginning and discuss applications related to dual canonical bases in Subsection 4.1.6 and $\mathfrak{sl}_n$-link polynomials in Section 4.2.

First, following an idea already given for $n = 3$ in [75], we show how to relate flows on $\mathfrak{sl}_n$-webs to standard fillings of n-multitableaux in Subsection 4.1.3 and the weight of flows to Brundan, Kleshchev and Wang's notion of degree for such n-multitableaux Subsection 4.1.4. That is, we show that all $\mathfrak{sl}_n$-webs with flows u_f can be obtained from standard fillings of n-multitableaux via an *extended growth algorithm*. We start by giving a method to turn a flow on a $\mathfrak{sl}_n$-web into such a filling. Then we give the inverse process: The *extended growth algorithm*. We show that they are inverses in Proposition 4.8 and then prove in Proposition 4.12 that the degree works out as well.

In addition we show in Lemma 4.9 how to generate all $\mathfrak{sl}_n$-webs $u \in W_n(\vec{k})$ from a suitable highest weight vector v_h using q-skew Howe duality and a sequence of $F_i^{(j)}$. We note that this shows that any reasonable basis of $W_n(\vec{k})$ is monomial, including e.g. Fontaine's basis [28].

Combining everything, we obtain in Theorem 4.15 an *algorithm* to evaluate closed $\mathfrak{sl}_n$-webs. A question, as pointed out by Cautis, Kamnitzer and Morrison in Section 1.5 and in the second remark after Lemma 2.2.1 in [20], that can not be done directly yet by using the $\mathfrak{sl}_n$-web relations. But the methods that we describe in this section turn out to be quite powerful as we illustrate later by giving the other two applications mentioned above.

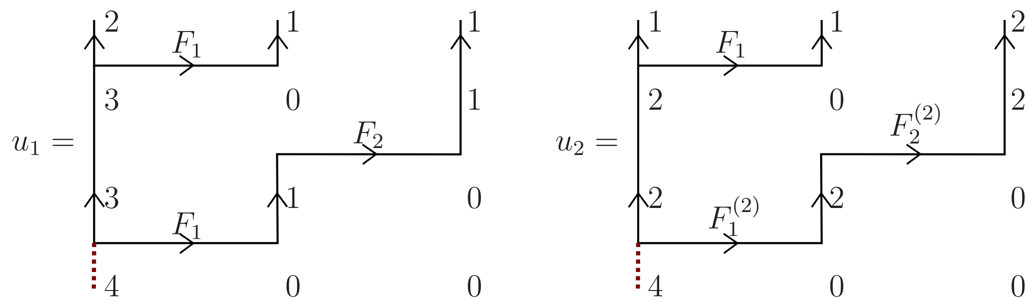
4.1.2. *Pictorial q-skew Howe duality: An example.* Before we start let us recall by an example how the translation of a string of $F_i^{(j)}$ acting on a highest weight vector v_h to a $\mathfrak{sl}_n$-web u works. The reader unfamiliar with this process, which is crucial for everything that follows, is encouraged to take a look at for example [20], [50] or [75] for a more detailed discussion.

Example 4.1. Let $n = 4$, $\ell = 1$ and let $v_h = v_{(4)}$ be the highest weight vector for the partition (4^1). It is worth noting that we use $\mathfrak{gl}_m$-weights when we picture q-skew Howe duality, i.e. one can read of the corresponding $\mathfrak{gl}_m$-weight $\vec{k}$ for a fixed level by taking the numbers in order from left to right as k_j. These can be turned into $\mathfrak{sl}_m$-weights by the rules in Equation 3.2.8.

Assume that we have the two stings

$$\mathrm{qH}(u_1) = F_1F_2F_1 \quad \text{and} \quad \mathrm{qH}(u_2) = F_1F_2^{(2)}F_1^{(2)}.$$

Then $\mathrm{qH}(u_{1,2})v_h$ will generate the following $\mathfrak{sl}_4$-webs u_1 and u_2 under q-skew Howe duality.



[8] Recall that we tend to, by abuse of notation, sometimes only call the generators of $\mathbf{Sp}_f^n(\mathbf{U}_q(\mathfrak{sl}_n))$ (i.e. no formal $\bar{\mathbb{Q}}(q)$-sums, but all possible pictures) $\mathfrak{sl}_n$-webs. We hope that the difference is clear from the context.

Note that these two $\mathfrak{sl}_4$-webs are not the same, since they have different labels (which can be read of from the number grid).

4.1.3. *The extended growth algorithm.* Denote by $W_n(\vec{k},\vec{S})$ the set of all possible $\mathfrak{sl}_n$-webs u that can be obtained by a string of divided powers of F acting on a highest weight vector $v_h = v_{(n^\ell)}$ (without taking any $\mathfrak{sl}_n$-web relations in account at the moment) together with a flow f on u with boundary datum $\vec{S}$.

It is worth noting, as we show later in Lemma 4.9, that the set $W_n(\vec{k},\vec{S})$ includes *all* $\mathfrak{sl}_n$-webs with boundary $\vec{k}$.

We start now by defining a map $\iota\colon W_n(\vec{k},\vec{S}) \to \mathrm{Std}(\vec{\lambda})$. We give it inductively using an *inductive algorithm*. The main idea of this process is simple: Assume that the k-th factor (in our notation read from right to left) of $\mathrm{qH}(u)$ is $F_{i_k}^{(j_k)}$. Then the k-th step of the algorithm should add j_k-nodes labeled k with residue i_k.

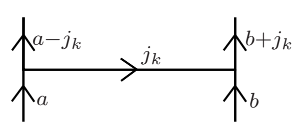
Recall we shift the residues of the corresponding n-multitableaux up by ℓ (the reader should compare this with our convention in Definition 3.1).

Definition 4.2. (**Flows to fillings**) Given a fixed pair $(\vec{k},\vec{S})$ and a $\mathfrak{sl}_n$-web $u_f \in W_n(\vec{k},\vec{S})$ and a string that generates u, i.e. $\mathrm{qH}(u) = F_{i_{m'}}^{(j_{m'})}\cdots F_{i_1}^{(j_1)}$.

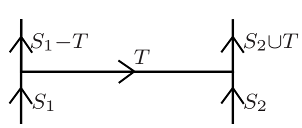
We associate to it *inductively* a standard n-multitableaux $\iota(u_f) \in \mathrm{Std}(\vec{\lambda})$ as follows (we note again that we always read the string of F's from right to left).

1. At the initial stage set $\vec{T}_0 = (\emptyset,\dots,\emptyset)$.
2. At the k-th step use $F_{i_k}^{(j_k)}$ and the local flow on the corresponding ladder to determine the operation performed on $\vec{T}_{k-1}$. We give the rule together with the operation
$$\mathbf{k}\colon \vec{T}_{k-1} \mapsto \vec{T}_k$$
below.
3. Repeat until $k = m'$.
4. Then set $\iota(u_f) = \vec{T}_{m'}$.

Assume that the ladder that corresponds to the k-th move $F_{i_k}^{(j_k)}$ is



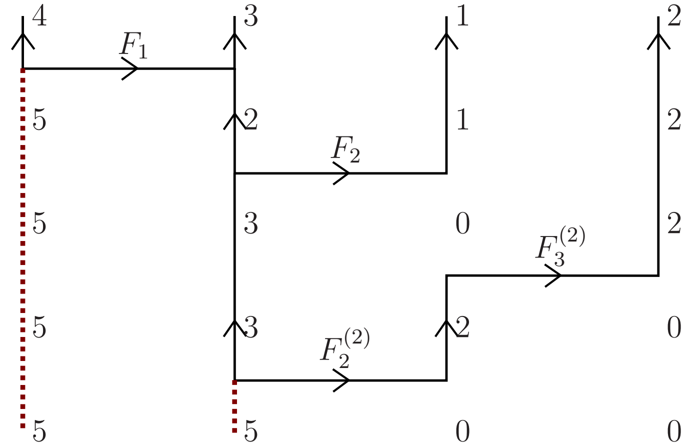
and the local flow on this ladder is



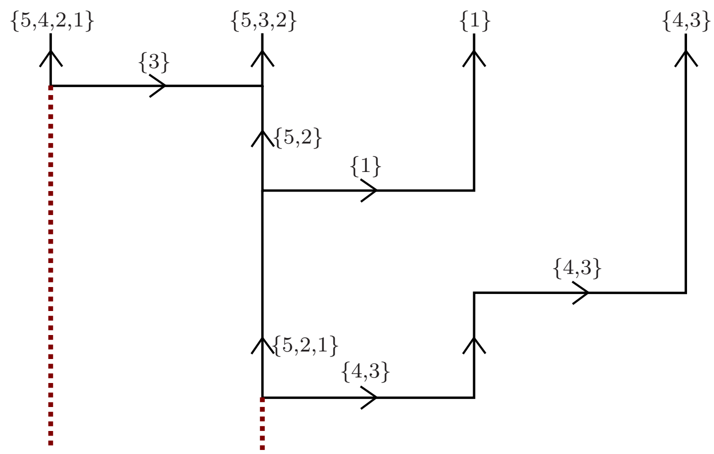
for suitable subsets $S_1,S_2,T \subset \{n,\dots,1\}$. The subset T will be, by our flow conventions, of the form $T = \{t_{j_k},\dots,t_1\}$ for $t_1 < \cdots < t_{j_k}$. Then $\mathbf{k}$ should add a node of residue i_k for all $t_{k'}$ to the $t_{k'}$-th part of $\vec{T}_k$ (recall that the parts of n-multitableaux are ordered from right to left as well).

Let us give an example before we show the non-trivial fact that the algorithm is well-defined.

Example 4.3. Given $n = 5$, $v_h = v_{(5^2)}$ and $\mathrm{qH}(u) = F_1F_2F_3^{(2)}F_2^{(2)}$ we obtain a $\mathfrak{sl}_5$-web u using q-skew Howe duality as follows.



Let us choose the following flow for it (here $\vec{S} = (\{5,4,2,1\},\{5,3,2\},\{1\},\{4,3\})$).



The algorithm makes five steps now, i.e. four honest ones corresponding to the four divided powers and an initial step. The steps are

$$\begin{aligned}\vec{T}_0 = (\,\emptyset\,,\,\emptyset\,,\,\emptyset\,,\,\emptyset\,,\,\emptyset\,) &\mapsto \vec{T}_1 = \left(\,\emptyset\,,\,\begin{array}{|c|}\hline 1\\\hline\end{array}\,,\,\begin{array}{|c|}\hline 1\\\hline\end{array}\,,\,\emptyset\,,\,\emptyset\,\right)\\ &\mapsto \vec{T}_2 = \left(\,\emptyset\,,\,\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\,,\,\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\,,\,\emptyset\,,\,\emptyset\,\right)\\ &\mapsto \vec{T}_3 = \left(\,\emptyset\,,\,\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\,,\,\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\,,\,\emptyset\,,\,\begin{array}{|c|}\hline 3\\\hline\end{array}\,\right)\\ &\mapsto \vec{T}_4 = \left(\,\emptyset\,,\,\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\,,\,\begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}\,,\,\emptyset\,,\,\begin{array}{|c|}\hline 3\\\hline\end{array}\,\right) = \iota(u_f).\end{aligned}$$

It is worth noting that the last step is a "blueprint" why this algorithm is well-defined, i.e. the corresponding new node has to be of residue 1 and there are two possibilities with no addable nodes of residue 1.

But these two cases can not occur if the flow at the upper middle upwards pointing edge is $\{5,2\}$, since the flows have to be disjoint. Only if one changes that local part of the flow, and therefore the former local parts too, the last step could be addition of such a node in the first or fourth entry of the last 4-multitableaux $\vec{T}_4$.

Lemma 4.4. *The algorithm of Definition 4.2 is well-defined. Moreover, we have*

$$\iota(u_f) = \iota(v_{f'}) \Leftrightarrow u = v \text{ and } f = f',$$

where the equality of $\mathfrak{sl}_n$-webs and flows is not taking any $\mathfrak{sl}_n$-web relations (including isotopies) into account.

Proof. All parts of the proof follow the same idea, i.e. we use induction on the total number $\ell(\mathrm{qH}(u))$ of $F_i^{(j)}$'s of the string of $F_i^{(j)}$'s that generate the $\mathfrak{sl}_n$-web u. The induction step is to remove the last, i.e. leftmost, factor $F_i^{(j)}$, to create a smaller $\mathfrak{sl}_n$-web $u^<$ for which the statement is already known by the hypothesis. To summarize assume that $\ell(\mathrm{qH}(u)) = r$. Then we have

$$u = F_{i_r}^{(j_r)} \prod_{k=1}^{r-1} F_{i_k}^{(j_k)} v_h \quad \text{and} \quad u^< = \prod_{k=1}^{r-1} F_{i_k}^{(j_k)} v_h.$$

Then we just check what the last step could do. It is worth noting that one has to check all cases of total length $\ell_{\mathrm{t}}(\mathrm{qH}(u)) = \sum j_k \leq n$, since the divided power can go up to n.

But that everything is well-defined follows for these cases, because all cases with total length $\leq n$ are just the first ladder steps given by $F_{i_1}^{(j_1)}$ which can not run into ambiguities, since we fill the empty n-multitableaux with at most n nodes and all of the correct residue due to our residue normalization. Moreover, the possible addable nodes of residue i_2 are given by $S^1_{i_2} - S^1_{i_2+1}$, where $\vec{S}^1$ is the flow at the top of the first ladder move.

Otherwise, assume that it is well-defined for $u^<$ and the possible addable nodes of residue i_r are given by $\vec{S}^<$. Observe now that the given flow on the middle edge of the ladder for $F_{i_r}^{(j_r)}$ is determined by the smaller one $f^<$ at the boundary of $u^<$. Moreover, by construction, it has to be disjoint to the two incoming flows at the boundary. That is, $T \subset S^<_{i_r} - S^<_{i_r+1}$.

This shows that the last step can perform a legal move and hence the algorithm is well-defined and gives a standard n-multitableaux. Moreover, the possible addable nodes will now be determined by $\vec{S}$.

That the algorithm gives different results for different $\mathfrak{sl}_n$-webs u, v or different flows f, f' on one $\mathfrak{sl}_n$-web u follows in the same vein, i.e. it is clear by construction that the first step will give a different result for different inputs. By induction, we then only have to ensure that the first place where either u and v are different or where f and f' are different gives a different result. The first follows directly, since already the boundary vectors $\vec{k}_u$ and $\vec{k}_v$ will be different for u and v and hence the whole shape will be different. The second follows because different flows with the same boundary datum have to be different on the middle edge of the last ladder. But in this case the rules tell us to place the new nodes in different parts of the n-multitableaux. $\square$

The whole procedure also works the other way around, that is, given a fixed n-multitableaux $\vec{T} \in \mathrm{Std}(\vec{\lambda})$, one can generate a $\mathfrak{sl}_n$-web $u_f \in W_n(\vec{k}, \vec{S})$ together with a flow on it as we describe now. We call this algorithm, by a slight abuse of notation, an *extended $\mathfrak{sl}_n$-growth algorithm*.

Definition 4.5. (**Extended $\mathfrak{sl}_n$-growth algorithm**) The *extended $\mathfrak{sl}_n$-growth algorithm* is

$$\mathrm{g}\colon \mathrm{Std}(\vec{\lambda}) \to W_n(\vec{k}, \vec{S}),$$

given inductively as follows.

Let $\vec{T} \in \mathrm{Std}(\vec{\lambda})$ be a standard n-multitableaux with nodes labeled from $1, \dots, s$. We assign to it a $\mathfrak{sl}_n$-web u given by a sequence of divided powers of $F_{i_k}^{(j_k)}$ (under q-skew Howe duality) by

$$u = \prod_{k=1}^{s} F_{i_k}^{(j_k)} v_{(n^\ell)},$$

where i_k is the residue of the node(s) with entry k and j_k is their multiplicity.

Denote for $k' = 0, \dots, s$ the $\mathfrak{sl}_n$-web $u^{k'}$ obtained by

$$u^{k'} = \prod_{k=1}^{k'} F_{i_k}^{(j_k)} v_{(n^\ell)}.$$

The flow f on u is given inductively starting with a flow f_0 on the $\mathfrak{sl}_n$-web u^0 that has only some leashes for entries with label n given by the full set $\{n, \dots, 1\}$ on all leashes and nothing else.

Assume $0 < k'$ and that the flow $f_{k'-1}$ on $u^{k'-1}$ is given. Then extend the flow to $f_{k'}$ on $u^{k'}$ by extending the flow $f_{k'-1}$ on $u^{k'-1}$ such that the horizontal line in the ladder corresponding to the last move given by $F_{i_{k'}}^{(j_{k'})}$ is labeled with the set

$$S = \{\epsilon_n, \dots, \epsilon_1\} - \{0\}, \epsilon_{\tilde{m}} = \begin{cases} \tilde{m}, & \text{if the number } k' \text{ appears in the } n\text{-multitableaux } T_{\tilde{m}}, \\ 0, & \text{else}. \end{cases}$$

Note that, if well-defined, this determines the labels on the two upper edges of the ladder. As a final stage set $u_f = u^s_{f_s}$.

It is again not obvious that this algorithm is well-defined. But before proving this in Lemma 4.7 we give an example.

Example 4.6. Given the 5-multitableaux

$$\vec{T} = (T_5, T_4, T_3, T_2, T_1) = \left(\emptyset\, ,\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\, ,\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1} \end{array}\, ,\ \emptyset\, ,\ \begin{array}{|c|}\hline 3 \\ \hline \end{array} \right),$$

which is $\vec{T}_4$ in Example 4.3, we see that the residue sequence (recall the shift of residues) is $r(\vec{T}) = (2, 3, 2, 1)$ and the entries in order appear with multiplicities $2, 2, 1, 1$. Hence, we get again $F_1 F_2 F_3^{(2)} F_2^{(2)}$ as the string of F's.

To see that the flow is also the same we proceed inductively. At the 0-th step we only have the leashes labeled with $\{5, 4, 3, 2, 1\}$. The first step has entries in T_3 and T_4. Therefore, the flow on the first horizontal edges is defined to be $\{4, 3\}$ which forces the outgoing upper left to be $\{5, 2, 1\}$.

We easily see that the next steps gives exactly the same result as in Example 4.3.

Lemma 4.7. *The algorithm of Definition 4.5 is well-defined. Moreover, we have*

$$\mathrm{forget}(\mathrm{g}(\vec{T})) = \mathrm{forget}(\mathrm{g}(\vec{T}')) \Leftrightarrow r(\vec{T}) = r(\vec{T}'),$$

where $\mathrm{forget}(\cdot)$ *forgets the flow line and*

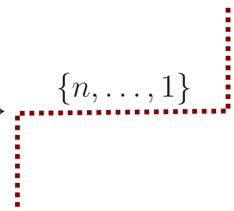
$$\mathrm{g}(\vec{T}) = \mathrm{g}(\vec{T}') \Leftrightarrow \vec{T} = \vec{T}',$$

where the equalities are again not taking any $\mathfrak{sl}_n$*-web relations (including isotopies) into account.*

Proof. The proof that the algorithm is well-defined and gives always different results for different n-multitableaux follows the same idea as in the proof of Lemma 4.4, i.e. induction on the length s of the n-multitableaux. We obtain $\vec{T}^{<}$ from $\vec{T}$ by removing all nodes with the highest entry such that the highest entry of $\vec{T}^{<}$ is $s-1$.

For both claims it is easy to verify all small cases, i.e. all cases with length $s=1$, by hand. Our residue convention ensures that the corresponding divided power does not kill the highest weight vector. Moreover, it is worth noting that a "full" n-multitableaux corresponds to a leash-shift with a "full" flow, that is

$$(\boxed{1},\dots,\boxed{1}) \mapsto$$



To see that the algorithm is well-defined note that we get a legal step from $\vec{T}^{<}$ to $\vec{T}$, i.e. a flow, because, if we add a ladder at the i-th position, then the values of S_i, S_{i+1} are determined by the same observation as above in the proof of Lemma 4.4. Moreover, to see that the string of $F_i^{(j)}$ does not kill the highest weight vector in the last step from $\vec{T}^{<}$ to $\vec{T}$, we note that the action of $F_{i_s}^{(j_s)}$ is determined by $\vec{k}^{<}$. And this is encoded in $\vec{T}^{<}$ by the residue sequence and multiplicities of the entries. If $F_{i_s}^{(j_s)}$ would kill the vector, then the configuration could not have been legal in the first place.

To see that n-multitableaux with a different residue sequence already give different $\mathfrak{sl}_n$-webs is because of the definition of the string of $F_i^{(j)}$'s. That different fillings give different flows follows, because the position of the nodes with the same label that are at different positions will give a different flow on the middle edge of the corresponding ladder.

On the other hand, that equal n-multitableaux give the same $\mathfrak{sl}_n$-webs with the same flow follows immediately and $r(\vec{T}) = r(\vec{T}')$ forces the underlying $\mathfrak{sl}_n$-webs to be the same follows because we obtain the string of $F_i^{(j)}$'s that generates the $\mathfrak{sl}_n$-webs only from the residue sequence. $\square$

Because the two algorithms given in the Definitions 4.2 and 4.5 are inverse procedures we note the following proposition. Moreover, since the $\mathfrak{sl}_n$-web is isotopy invariant, we obtain the same for the "rigid" n-multitableaux framework.

Proposition 4.8. *We have*

$$\iota \circ \mathrm{g} = \mathrm{id}_{\mathrm{Std}(\vec{\lambda})} \quad \textit{and} \quad \mathrm{g} \circ \iota = \mathrm{id}_{W_n(\vec{k},\vec{S})}.$$

Where we again not taking any $\mathfrak{sl}_n$-web relations (including isotopies) into account.

Proof. We use the two Lemmata 4.4 and 4.7, i.e. scrutiny of the inductive steps given in Definitions 4.2 and 4.5 shows that they reverse each other. We leave the details to the reader. $\square$

We illustrate now in an important lemma how one can write any $\mathfrak{sl}_n$-web $u \in W_n(\vec{k})$ *explicitly* as a string of $F_i^{(j)}$'s. In fact, our statement is a little bit stronger, since we allow *any* $\mathfrak{sl}_n$-web, e.g. also elliptic $\mathfrak{sl}_3$-webs in the $\mathfrak{sl}_3$ case, starting from the same highest weight vector v_h. This is important for example for the connection to the $\mathfrak{sl}_n$-link polynomials that we discuss in Section 4.2. We stress that Lemma 4.9 gives rise to an algorithm to obtain the string of $F_i^{(j)}$'s.

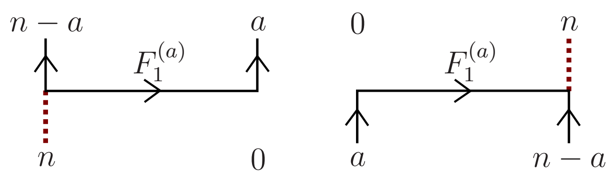
Lemma 4.9. *Any $u \in W_n(\vec{k}) \subset W_n(\Lambda)$, for all $\vec{k}$, can be written, using q-skew Howe duality, as*

$$u = \prod_{k=1}^{s} F_{i_k}^{(j_k)} v_{(n^\ell)}$$

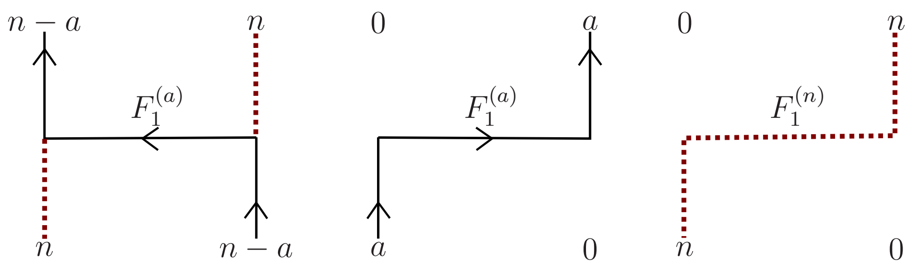
for some $s \in \mathbb{N}$. Moreover, this can be done in such a way that none of the $F_i^{(j)}$'s connects two nested and before the action of the $F_i^{(j)}$ unconnected components.

Proof. We prove the first statement by induction on the number of vertices of the $\mathfrak{sl}_n$-webs u. We use 1 here as the position index without loss of generality.

If u has no vertices at all, then we see that we have to check exactly five cases, i.e. cup and cap

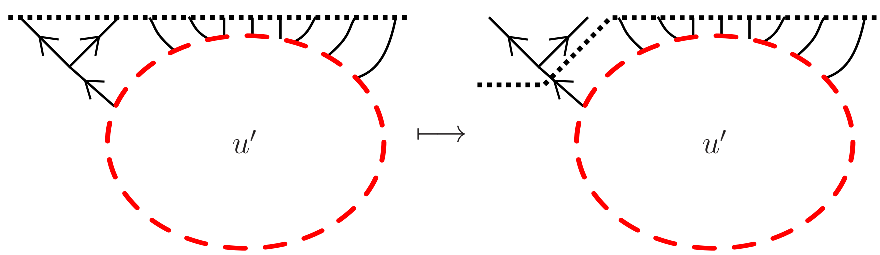


and three shifts, i.e. the left, right and the empty shift

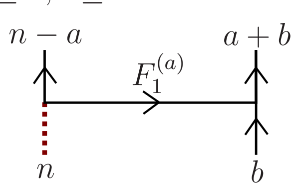


Here we can use any $0 \leq a \leq n$. This shows that any $\mathfrak{sl}_n$-web with no vertices can be obtained from $v_{(n^\ell)}$ by an explicit sequence of $F_i^{(j)}$'s starting from a suitable weight at the bottom which can be chosen as a highest weight in the closed cases.

Now assume that u has at least on vertex. Take the leftmost of the vertices of u with two outgoing edges (including leashes) that connects to the cut-line. Cut it away by changing the cut-line a little bit as illustrated below. The boundary data changes accordingly (we allow an arbitrary, finite number of 0's to the left).

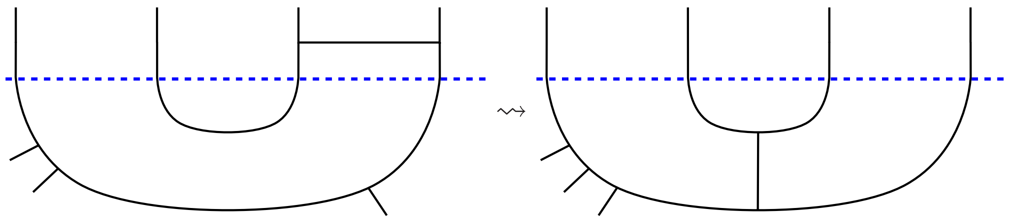


Since u' has fewer vertices than u, we can use induction and the observation that the last step can be realized as an $F_i^{(j)}$ depending on how we read the tripod, e.g. (the reader is encouraged to check the other possibilities) for suitable $0 \leq a, b \leq n$



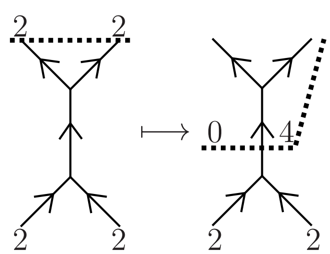
Hence, u can be realized as a string of suitable chosen $F_i^{(j)}$'s. It should be noted that the highest weight vector stays the same, since it is located at the bottom of the picture and the number of its entries is fixed by $\vec{k}$. Moreover, a case by case check for all possible boundary data reveals that the case of "missing" leashes does not appear.

To see the second statement we note that we can freely use isotopies as illustrated below.

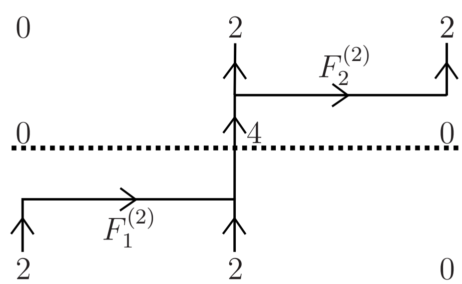


That is, we can always avoid to connect nested parts by shifting the $F_i^{(j)}$-ladder around. Note that such a procedure could require a longer string of $F_i^{(j)}$'s (one has to be careful how to read these pictures, but we hope that they illustrate that such a situation can always be avoided). $\square$

Example 4.10. For example a $\mathfrak{sl}_n$-web u with a local dumbbell and $n > 4$



can be realized as (for simplicity with $1, 2$ as position indices)



Thus, in the notation of Lemma 4.9, the $\mathfrak{sl}_n$-web u' has a $F_1^{(2)}$ as a leftmost factor in its product of $F_i^{(j)}$'s. Hence, we have

$$u' = F_1^{(2)} \prod_k F_{i_k}^{(j_k)} v_h \rightsquigarrow u = F_2^{(2)} F_1^{(2)} \prod_k F_{i_k}^{(j_k)} v_h.$$

As another example we encourage the reader to verify is that the $\mathfrak{sl}_4$-web u from Example 3.26 can be generated by

$$u = F_7^{(2)} F_3 F_1 F_2 F_1 F_3 F_4^{(2)} F_3^{(2)} F_4 F_5 F_4 F_2 F_1 F_3^{(2)} F_2^{(4)} F_6^{(4)} F_5^{(4)} F_4^{(4)} F_3^{(4)} v_{(4^3)}.$$

If we use this string to generate the $\mathfrak{sl}_4$-web u, then the flow f from Example 3.26 will be converted to the following 4-multitableau.

$$\iota(u_f) = \left(\begin{array}{ccccc} 1&2&3&4&19 \\ 5&6&9&10& \\ 15&16&18&& \end{array} , \begin{array}{cccc} 1&2&3&4 \\ 5&6&11& \\ 7&8&14& \end{array} , \begin{array}{ccccc} 1&2&3&4&19 \\ 5&16&17&& \end{array} , \begin{array}{cccc} 1&2&3&4 \\ 5&12&13& \\ 17&&& \end{array} \right)$$

The reader is invited to check that the degree of this 4-multitableau is 9, i.e. exactly the weight. We see in Proposition 4.12 that this is in fact no coincidence.

The Proposition 4.8 together with Lemma 4.9 imply that any "reasonable" basis of the $\mathfrak{sl}_n$-web space $W_n(\vec{k})$ is *monomial*, i.e. given by a sequence of $F_i^{(j)}$'s acting on a highest weight vector v_h. In fact, given a spanning set of $\mathfrak{sl}_n$-webs of $W_n(\vec{k})$, the hardest part is to show linear independence.

Some "reasonable" bases of $W_n(\vec{k})$ are the basis given by all $\mathfrak{sl}_2$-arc diagrams (here $n = 2$), Kuperberg's basis of non-elliptic $\mathfrak{sl}_3$-webs (here $n = 3$), intermediate crystal bases in the sense of Leclerc and Toffin [46] under q-skew Howe duality (see [75] or [50]) and Fontaine's basis [28].

Corollary 4.11. *All of the bases of $W_n(\vec{k})$ mentioned above are monomial.* □

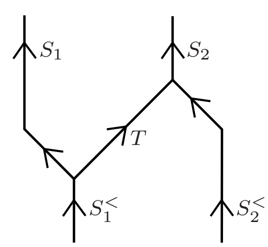
4.1.4. *BKW's degree and the weight of flows.* We are going to show now that the result of Proposition 4.8 can be strengthened. To be more precise, both $W_n(\vec{k})$ and $\mathrm{Std}(\vec{\lambda})$ are graded. The first one by the weight of the flows and the second one by Brundan, Kleshchev and Wang's degree for multitableaux.

Proposition 4.12. *Both maps*

$$\iota\colon W_n(\vec{k}, \vec{S}) \to \mathrm{Std}(\vec{\lambda}) \quad \textit{and} \quad \mathrm{g}\colon \mathrm{Std}(\vec{\lambda}) \to W_n(\vec{k}, \vec{S})$$

preserve the degree.

Proof. First lets us take a look how to read of the weight for a ladder. Assume that the flow on the top of a ladder is given by $\vec{S} = (S_1, \dots, S_m)$, at the bottom by $\vec{S}^< = (S_1^<, \dots, S_m^<)$ and at its horizontal edge by T. Moreover, assume for simplicity that the ladder comes from an action of F_1, i.e. that it is a ladder at position 1. Then, by our convention how to draw ladders, we have



The weight $\mathrm{wt}(u)$ is now given by $\ell(S_1, T) - \ell(T, S_2^<)$, that is, by counting how many pairs of the set $T \times S_2^<$ are strictly ordered and subtract the number of strictly ordered pairs of $S_1 \times T$. Since $S_1 = S_1^< \cup T$, this is the same as

$$\mathrm{wt}(u) = \ell(S_1, T) - \ell(T, S_2^<) = \ell(S_1^<, T) - \ell(T, S_2^<) - \frac{1}{2}|T|(|T| - 1). \tag{4.1.1}$$

We are going to show that the map ι preserves the degree. The other direction follows in a similar vein, since both algorithm are inverses.

To proof that ι preserves the degree we can use a similar induction as in the Lemmata 4.4 and 4.7 before. One easily verifies that the small cases, i.e. the empty shift and all possible flows on caps

and cups, preserve the degree. It is worth noting that the shift of the degree

$$a = \sum_{i=0}^{N^j-1} i \tag{4.1.2}$$

from Definition 3.5 is exactly the shift by $\frac{1}{2}|T|(|T|-1)$, because $|N^j| = |T|$. For example, if the first step is an empty shift, then $S_1^< = T = \{n, \dots, 1\}$ and $S_2^< = \emptyset$ which gives the desired answer.

For a $\mathfrak{sl}_n$-web with a flow u_f and $\iota(u_f) = \vec{T}$, we can assume that the degree is preserved for $u_{f^<}^<$. Hence, we only have to verify that the degree is still preserved in the last step of the algorithm. To see this we note that the three terms $\ell(S_1^<, T)$, $\ell(T, S_2^<)$ and $\frac{1}{2}|T|(|T|-1)$ from Equation 4.1.1 are exactly the three numbers from Definition 3.5, i.e.

$$\ell(S_1, T) = |\mathsf{A}^{k \succ N}(\vec{T}^j)|, \ell(T, S_2^<) = |\mathsf{R}^{k \succ N}(\vec{T}^j)| \text{ and } \frac{1}{2}|T|(|T|-1) = a.$$

This finishes the proof, since both, the (total) weight wt and $\deg_{\mathrm{BKW}}$ are locally the same and are both defined inductively. $\square$

4.1.5. *The evaluation algorithm.* We conclude this part by giving an algorithm to evaluate all closed $\mathfrak{sl}_n$-webs w. It is worth saying again that this is non-trivial for $n > 3$ since we have relations as the square-switch 3.2.7. In order to show that the algorithm really gives the desired answer we have to use all the observations from this section, i.e. Lemma 4.9 to write a closed $\mathfrak{sl}_n$-webs w as a string of $F_i^{(j)}$'s, the conversion from this to n-multitableaux given in Definition 4.2 and the Proposition 4.12, together with the interpretation via intertwiners from Section 3.2.

It is worth noting that the algorithm below, in a slightly re-arranged form, works for any $\mathfrak{sl}_n$-web $u\colon \emptyset \to \Lambda^{\vec{k}}\bar{\mathbb{Q}}^n$. In this case the algorithm does not give a quantum number, but the decomposition in terms of elementary tensors as in Equation 3.2.14.

Definition 4.13. (Evaluation of $\mathfrak{sl}_n$-webs) Given a $\mathfrak{sl}_n$-web $u \in W_n(\vec{k}) \cong \mathrm{Inv}_{\dot{\mathbf{U}}_q(\mathfrak{sl}_n)}(\Lambda^{\vec{k}}\bar{\mathbb{Q}}^n)$ together with a sequence of generating $F_i^{(j)}$'s, i.e.

$$u = \prod_{k=1}^{s} F_{i_k}^{(j_k)} v_{(n^\ell)},$$

we assign to it a set $\mathrm{ev}_u = \{\vec{T}_1, \dots, \vec{T}_a\}$ of standard n-multitableaux $\vec{T}_b$ *inductively* as follows.

(1) Set $\mathrm{ev}_u^0 = \{\emptyset\}$, where $\emptyset$ denotes the empty n-multitableaux.
(2) In each step $1 \leq k \leq s$ add certain (explained below) new n-multitableaux $\vec{T}^k$ to ev_u^{k-1} and obtain a new set ev_u^k.
(3) After each step $1 \leq k \leq s$ remove all old n-multitableaux $\vec{T}^{k-1}$ from ev_u^k.
(4) Repeat (2)+(3) until $k = s$. Set $\mathrm{ev}_u = \mathrm{ev}_u^s$.

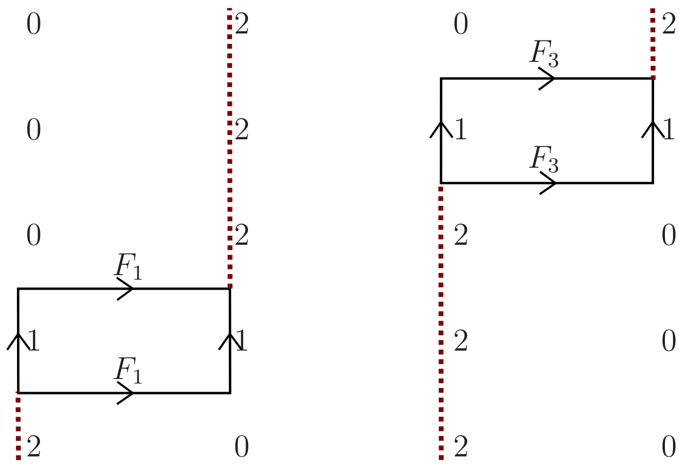
The way to decide which n-multitableaux $\vec{T}^k$ should be added in the k-th step is to take all possible ways to add j_k nodes with residue i_k labeled k to a $\vec{T}^{k-1}$ such that the result is again a standard n-multitableaux. Do this for all possible $\vec{T}^{k-1}$.

The *evaluation* of a closed $\mathfrak{sl}_n$-web $w \in \mathrm{End}_{\dot{\mathbf{U}}_q(\mathfrak{sl}_n)}((n^\ell))$ is

$$\mathrm{ev}(w) = \sum_{\vec{T} \in \mathrm{ev}_w} q^{\deg_{\mathrm{BKW}}(\vec{T})} \in \mathbb{N}[q, q^{-1}].$$

It is again not immediately clear why the evaluation algorithm gives the right answer. Moreover, it is not clear, why it is well-defined (independent of choices) at all. Before we show that this is indeed the case, let us give an example.

Example 4.14. In order to get started we take a (very) small, but hopefully illustrating, example. Consider two circles as a $\mathfrak{sl}_2$-web w in the $\mathfrak{sl}_2$ case. We know in this case that the evaluation should give $[2]^2 = q^2 + 2 + q^{-2} \in \mathbb{N}[q, q^{-1}]$. We can write it as a string of $F_i^{(j)}$ as follows.



Hence, because we also have an empty shift at the bottom (note that we usually do not perform the last steps at the top to re-order to a lowest weight since the corresponding weight modules are isomorphic anyway. The same is true for the bottom of course, but, due to our convention, we need the extra nodes such that the placement works in the way we stated it above. But empty shifts never do anything interesting), we have $F_3F_3F_1F_1F_2^{(2)}$ for w. Recall that we have a shift of residues given by the number of 2's at the bottom. From the algorithm in Definition 4.13 we get the four 2-multitableaux

$$\vec{T}_1 = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 5 \\ \cline{1-1} \end{array} \right) \text{ and } \vec{T}_2 = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 5 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \right)$$

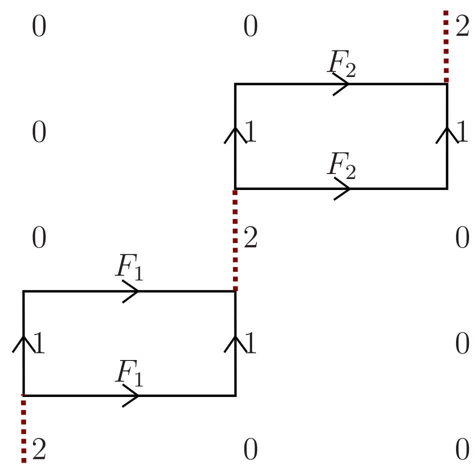
and

$$\vec{T}_3 = \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 5 \\ \cline{1-1} \end{array} \right) \text{ and } \vec{T}_4 = \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 5 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1} \end{array} \right),$$

because in the first step (the one for $F_2^{(2)}$) we have exactly one option where we can add two nodes with residue 2 to the empty 2-multitableaux. Then we have two choices to add nodes for the two

F_1's and the same happens for the two F_2's. The reader should check that the degrees for the 2-multitableaux from $\vec{T}_1$ to $\vec{T}_4$ are $2, 0, 0, -2$. These are exactly the powers of the q's in $[2]^2$.

Note that the way to obtain w as a string of $F_i^{(j)}$ is far from being unique. For example

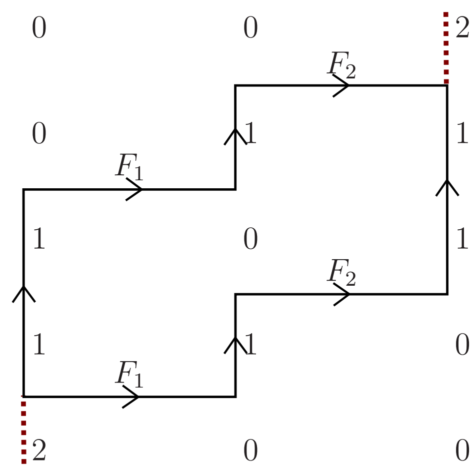


The reader is encouraged to check that the result is again a set of four 2-multitableaux of the right degree. This time they are

$$\vec{T}_{1,2} = (\,\boxed{1\,|\,-}\,,\ \boxed{2\,|\,-}\,) \text{ or } \vec{T}_{3,4} = (\,\boxed{2\,|\,-}\,,\ \boxed{1\,|\,-}\,),$$

where the $-$ should be filled with either 3 in the first and 4 in the second or vice versa.

A crucial difference (also from the viewpoint of the $\mathfrak{sl}_n$-link polynomials) is to change the sequence for the two circles $w = F_2F_2F_1F_1$ to $w' = F_2F_1F_2F_1$. This gives the following $\mathfrak{sl}_2$-web.



The algorithm gives now only the two 2-multitableaux

$$\vec{T}_1 = (\,\boxed{1\,|\,2}\,,\ \boxed{3\,|\,4}\,) \text{ or } \vec{T}_2 = (\,\boxed{3\,|\,4}\,,\ \boxed{1\,|\,2}\,),$$

because the nodes with labels 2 and 3 switch their residue. The two 2-multitableaux are of degree 1 and -1 giving the evaluation $q + q^{-1} = [2] \in \mathbb{N}[q, q^{-1}]$ as expected.

Theorem 4.15. *The evaluation of $\mathfrak{sl}_n$-webs is independent of the choices involved. Moreover, for any two $\mathfrak{sl}_n$-webs $u, v \in W_n(\vec{k})$ the evaluation in the Definitions 3.22 and 4.13 satisfy ($w = v^*u$)*

$$\mathrm{ev}(v^*u) = \sum_{\vec{T}\in \mathrm{ev}_w} q^{\deg_{\mathrm{BKW}}(\vec{T})} = q^{-d(\vec{k})}\langle u, v\rangle_{\mathrm{Kup}} = q^{-d(\vec{k})}\langle u, v\rangle_{\mathrm{Shap}},$$

i.e. the evaluation using n-multitableaux gives (up to a shift by $-d(\vec{k})$), the Kuperberg bracket $\langle \cdot, \cdot\rangle_{\mathrm{Kup}}$ which is also the q-Shapovalov form $\langle \cdot, \cdot\rangle_{\mathrm{Shap}}$.

Proof. To prove that the algorithm is well-defined we observe that the procedure is deterministic, i.e. the algorithm itself can not run into ambiguities.

To see that it is independent of the involved choices note that the algorithm is just a way to find possible flow lines on u under the interpretation given in Definition 4.2. That it is independent of the choices, i.e. how to write a certain local move, and isotopies follows now from the Lemmata 4.4 and 4.7. To be more precise, if we start with two different n-multitableaux that correspond to the same flow on a fixed $\mathfrak{sl}_n$-web u (including isotopies). Then we can convert both to the $\mathfrak{sl}_n$-web framework and we can use the isotopy invariance to see that they agree.

That it is also independent of the highest weight vector follows from Theorem 3.28 and the observation that we have normalized the degree in such a way that all empty shifts are of degree zero. Hence, since tensor products of trivial representation have an, up to a scalar, unique basis vector, the Theorem 3.28 and our normalization imply that the resulting evaluation $\mathrm{ev}(u)$ is a fixed element in $\mathbb{N}[q, q^{-1}]$.

The third equality is a consequence of Proposition 3.24. Hence, it only remains to show the second equality. This equality can be proven using Theorem 3.28 again.

That is, one needs to show that the coefficients in the relations given in Definition 3.12 are exactly given by the weight of the local flows. Furthermore, one has to take the change of $\vec{k}$ into account to see how the shift $d(\vec{k})$ changes stepwise. This is a straightforward, but exhausting, calculation and we do not do it here (although, because of the Lemmata 4.4 and 4.7, we do not have to check the isotopy relations). For example, if $n = 3$, then a closed circle (i.e. 3.2.4 with $a + b = 3$) has three flows of degree $2, 0, -2$ giving $q^2 + 1 + q^{-2} = [3]$. □

4.1.6. *An application: Dual canonical bases and $\mathfrak{sl}_n$-webs.* As an application of Theorem 4.15 we will conclude this section by giving an explicit and algorithmic iff-condition for a $\mathfrak{sl}_n$-web u to be dual canonical. Dual canonical for $\mathfrak{sl}_n$-webs means canonical on the q-skew Howe dual side, see e.g. Corollary 4.21 in [50]: Thus, in our notation, having *positive* exponent properties.

We do not recall the definition of the *lower global crystal basis* (in the sense of Kashiwara), which is sometimes also called *canonical basis* (in the sense of Lusztig), of the $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-module of highest weight Λ consisting of $\mathfrak{sl}_n$-webs that we already mentioned before and denote by $W_n(\Lambda)$. We are seeing it as a $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-module in the following. It is worth noting that this works in a more general framework, but we are mostly interested in the ones of highest weight Λ.

The ready who is interested in a more detailed discussion about these bases can check for example [6], [11] or Lusztig's book [49] and a discussion related to $\mathfrak{sl}_n$-webs can be found in [50].

Recall that there is a unique q-antilinear *bar-involution* ϕ on $W_n(\Lambda)$ determined by $\phi(v_\Lambda) = v_\Lambda$ and $\phi(Xv_\Lambda) = \overline{X}v_\Lambda$ for a vector v_Λ of highest weight Λ and any $X \in \dot{\mathbf{U}}_q(\mathfrak{sl}_m)$. We can use the

q-Shapovalov form $\langle\cdot,\cdot\rangle_{\mathrm{Shap}}$ on $W_n(\Lambda)$ (see e.g. [55] before Corollary 4.10) to define *Lusztig's symmetric bilinear form* by setting $(\cdot,\cdot)_{\mathrm{Lusz}} = \overline{\langle\cdot,\phi(\cdot)\rangle}_{\mathrm{Shap}}$.

Moreover, it is known that $W_n(\Lambda)$ is *parametrized* by semi-standard tableaux of shape (n^ℓ), which we denote by $\mathrm{Std}^s((n^\ell)) \subset \mathrm{Col}((n^\ell))$. For a column-strict tableaux T we can define the *column-word* $co(T) = (c_1, \dots, c_{n\ell})$ to be a sequence of the entries of the columns of T read from top to bottom and then from left to right. Note that this sequence has length $n\ell$. Then the set $\mathrm{Col}((n^\ell))$ is partial order by

$$T \leq T' \Leftrightarrow c(T') - c(T) \in \mathbb{N}^{n\ell} \ \text{ with } \ c(T^{(\prime)}) = (c_1^{(\prime)}, c_1^{(\prime)} + c_2^{(\prime)}, \dots, c_1^{(\prime)} + \cdots + c_{n\ell}^{(\prime)}).$$

Since we tend to use n-multipartitions and n-multitableaux instead let us state what this means in our notation. A column-strict tableaux T of shape (n^ℓ) corresponds to a n-multipartition $\vec{\lambda}$ by subtracting from each row the row number and obtain a new column-strict tableaux $\tilde{T}$. Read the k-th column from bottom to top to obtain in this way the $n+1-k$-th partition λ_{n+1-k} of the $\vec{\lambda} = (\lambda_n, \dots, \lambda_1)$. It is easy to see that this process is in fact invertible (the usage $n+1-k$ instead of k due to our reading convention for n-multipartitions).

Write $\vec{\lambda}_T$ for the corresponding n-multipartition. Then $T \leq T'$ iff $\vec{\lambda}_T \trianglelefteq \vec{\lambda}_{T'}$, where $\trianglelefteq$ is the dominance order from Definition 3.7. As a small example consider the following.

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \leq \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \text{and} \quad \left(\emptyset, \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}\right) \trianglelefteq \left(\begin{array}{|c|}\hline \\ \hline \end{array}, \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \end{array}\right).$$

Note that the conversion of a column-strict tableaux T to a $\vec{S} = (S_1, \dots, S_k)$ is given by counting the multiplicities of the entry r and obtain an r-element subset $S_r \subset \{n, \dots, 1\}$ by taking the column numbers in which the entry appears as elements of S_r. Our Proposition 4.8 is actually stronger: For each boundary condition $\vec{S}$ there exists a $\mathfrak{sl}_n$-web u_f that realizes this condition. To see this note that, as explained above, one can covered $\vec{S}$ to a n-multipartition $\vec{\lambda}$, then fill $\vec{\lambda}$ in *any* standard way and use Proposition 4.8 to generate a $\mathfrak{sl}_n$-web u_f. Thus, it makes sense to write x_T since this corresponds 1:1 to the elementary tensors $x_{\vec{S}}$ from Subsection 3.2.2.

A standard argument that works in more generality shows that a canonical basis, *if it exists*, is unique for a given pre-canonical structure. For a more general discussion see e.g. [76]. Moreover, Lusztig and Kashiwara proved that there exists a canonical basis $\{b_T \mid T \in \mathrm{Std}^s((n^\ell))\}$ of $W_n(\Lambda)$ with respect to the pre-canonical structure given by the elementary tensors $\{x_T \mid T \in \mathrm{Col}((n^\ell))\}$, the bar-involution ϕ and Lusztig's symmetric bilinear form $(\cdot,\cdot)_{\mathrm{Lusz}}$.

In order to state the condition we need to extend the notion of a *"canonical flow"* f_c for a fixed $\mathfrak{sl}_n$-web $u \in W_n(\vec{k})$. To understand the notion recall that, e.g. by Lemma 4.9, any $\mathfrak{sl}_n$-web u can be obtained from a string of $F_i^{(j)}$'s acting on a suitable highest weight vector v_h. While the elements of $W_n(\Lambda)$ are indexed by semi-standard tableaux of shape (n^ℓ), the elements of the tensor product $\Lambda^{k_1}\bar{\mathbb{Q}}^n \otimes \cdots \otimes \Lambda^{k_m}\bar{\mathbb{Q}}^n$ are indexed by column-strict tableaux of shape (n^ℓ) and $W_n(\Lambda)$ is a direct summand of it. Let us denote by $\mathrm{sh} \in \mathbb{Z}$ some shift. Then Theorem 3.28 says that

$$u = q^{\mathrm{sh}} x_T + \sum_{T \leq T'} c(u,T') x_{T'}, c(u,T') \in \mathbb{N}[q,q^{-1}], T, T' \in \mathrm{Col}((n^\ell)) \tag{4.1.3}$$

$$= q^{\mathrm{sh}} x_{\vec{\lambda}_T} + \sum_{\vec{\lambda}_T \trianglelefteq \vec{\lambda}_{T'}} c(u,\vec{\lambda}_{T'}) x_{\vec{\lambda}_{T'}}, c(u,\vec{\lambda}_{T'}) \in \mathbb{N}[q,q^{-1}], \vec{\lambda}_T, \vec{\lambda}_{T'} \in \Lambda^+(c(\vec{\lambda}_{T^{(\prime)}}), c(\vec{k}), n). \tag{4.1.4}$$

We do not have a positive exponent property in general. Note that we are mostly interested in the case when the inequalities are strict and the leading coefficient is 1, because it is one condition for a vector to be (dual) canonical.

By Theorem 3.28 the flows encode the coefficients of u in terms of elementary tensors. The "canonical flow" now should be the flow that encodes the leading coefficient in the decomposition above. Recall from the previous sections that a flow f can be translated to a string $\vec{S}_f$ of elements of $\mathfrak{P}(\{n,\ldots,1\})$ by looking at the boundary and to a n-multipartition $\vec{\lambda}_f$ by removing all numbers from its corresponding n-multitableaux $\vec{T}_f$ from Section 4.1.

It is very important in the following that we assume that the strings that generate our $\mathfrak{sl}_n$-webs are *not* arbitrary, but in such a way that they do not connect nested, unconnected components. This is always possible as explained in Lemma 4.9.

Definition 4.16. (**Canonical flow**) Given a $\mathfrak{sl}_n$-web u and a sequence of $F_i^{(j)}$'s generating u. The *canonical flow* f_c for u is the flow that corresponds (under Proposition 4.8) to the n-multitableaux $\vec{T}_c$ obtained inductively by placing j_k nodes with residue i_k in the rightmost possible position. We denote the corresponding n-multipartition by $\vec{\lambda}_c$.

It is worth noting that it is rather surprising that such a "greedy-like" algorithm is well-defined and gives an interesting result. We should note that the canonical flow will not be of degree zero in general, i.e. there can be some shift by sh. But, as we show below, it will always be of degree $\deg_{BKW}(u_{f_c}) \leq 0$. The proof itself is quite technical.

Lemma 4.17. *Given a fixed $\mathfrak{sl}_n$-web u. Then the canonical flow f_c on u exists, i.e. the algorithm from Definition 4.16 is well-defined.*

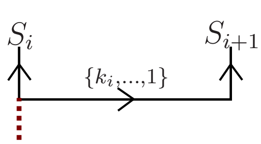
Moreover, $\deg_{\mathrm{BKW}}(\vec{T}_c) = \mathrm{wt}(u_{f_c}) = \mathrm{sh}$ *for some constant* $\mathrm{sh} \leq 0$ *and for all flows f on u the corresponding $\vec{\lambda}_f$ are bigger in the dominance order.*

Hence, the $\vec{\lambda}_c = \vec{\lambda}_T$ and sh *is the shift from Equation 4.1.3. This inequality is strict iff* $\mathrm{sh} = 0$.

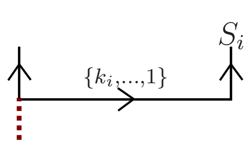
Proof. That the algorithm is well-defined, i.e. in each step one can place the correct number of nodes at the correct positions, follows by induction on the number of vertices $V(u)$ again. The induction step is as before removing the last $F_i^{(j)}$ of the string that generates u. Then it is true for $u^<$ and we can check locally that it still works.

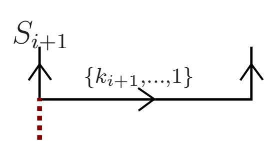
In fact, we prove something stronger. Recall that u has a boundary string $\vec{k} = (k_1,\ldots,k_m)$ and $\vec{S}_{u_c} = (S_1,\ldots,S_m)$ denotes the boundary of the canonical flow on u (if it exists) and the S_i are subsets of $\{n,\ldots,1\}$. We show that $|S_i - S_{i+1}| < \min(k_i, n-k_{i+1})$ iff S_k and S_{k+1} are not connected and belong to two nested components of u. Moreover, we also want to show at the same time that u has a canonical flow in the sense of Definition 4.16.

First we note that we are only interested in the boundary, that is we can ignore internal closed components and that the statement is certainly true for all shifts. So let u be a collection of arcs, i.e. $V(u) = 0$. We have to check three cases. The first two are

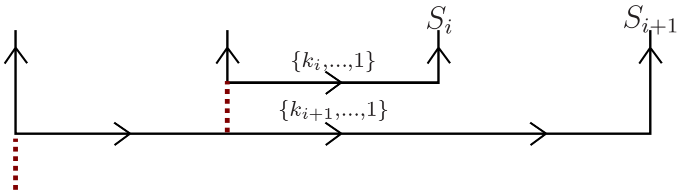


and





and the third is



In all these cases the canonical flow is displayed above. Hence, the canonical exists and satisfies the extra condition from above (recall that leashes have flow $\{n,\dots,1\}$ which splits into two disjoint flows at the top). Note that $\{k_i,\dots,1\}-\{n,\dots,n-k_{i+1}+1\}=\{\min(k_i,n-k_{i+1}),\dots,1\}$.

Moreover, that the statement is true if u has exactly one vertex follows in the same fashion by checking three extra cases involving a component that looks like a theta-web (we need this case too, because a ladder can have two vertices).

The main observation now is that one can always apply every non-killing divided power of F_i in the first two cases and the canonical flow will carry over, but one could run into problems in the last case.

Now assume $|V(u)|>1$. Remove the last ladder from u and obtain a $\mathfrak{sl}_n$-web $u^<$. Note that it is clear by the case-by-case check above that the statement will carry over from $u^<$ to u if this last ladder was an arc. Thus, we can freely assume that the last ladder has at least one vertex and we can use the induction hypothesis on $u^<$. But then the statement follows also for u, since we know by Lemma 4.9 that the last $F_i^{(j)}$ does not connect nested, unconnected components of $u^<$. But then, since the last $F_i^{(j)}$ does not kill $u^<$, we can apply the procedure from Definition 4.16 to the canonical flow on $u^<$, because of the translation between flows and n-multitableaux from Section 4.1. Moreover, the other statement also carries over. Thus, the algorithm is well-defined.

We observe that the second statement can in fact be strengthen. That is, each local step is of degree lower or equal zero (and therefore of course also the total result). To see this note that if a step would have addable nodes of the same residue to the right, then we would have placed them differently. Thus, the only contributions to the degree comes from removable nodes which always lower the degree and the total degree will be some constant $\mathrm{sh}\leq 0$.

That all other flows give bigger n-multipartitions follows immediately from the definition of the dominance order, since we place the nodes in the rightmost possible positions. But in general there can be non-canonical flows f with the same n-multipartition $\vec{\lambda}_f=\vec{\lambda}_{f_c}$, e.g. if u has a connected, internal, closed $\mathfrak{sl}_n$-web as for example a closed circle.

But if $\mathrm{sh}=0$, then this inequality has to be strict. This follows because the residue sequence of the n-multitableaux $\vec{T}$ have to be the same for all flows on u. That is $\vec{\lambda}_f=\vec{\lambda}_{f_c}$ and $f\neq f_c$ implies the existence of removable nodes, because $f\neq f_c\Leftrightarrow \vec{T}_f\neq\vec{T}_{f_c}$ and, by the argument above, $\vec{T}_{f_c}$ does not have addable nodes. But then $\mathrm{sh}<0$.

In the same vein, if $\mathrm{sh}<0$, then the existence of removable nodes allows use to define another n-multitableaux $\vec{T}_f\neq\vec{T}_{f_c}$ with $\vec{\lambda}_f=\vec{\lambda}_{f_c}$ by switching the corresponding entries of the nodes. $\square$

Example 4.18. The reader is invited to check that our notion of canonical flow for arc-diagrams in the case $n=2$ gives counter-clockwise oriented circles in the notation of Brundan and Stroppel [9] and in the case $n=3$ our definition gives exactly Khovanov and Kuperberg's notion of canonical flows for non-elliptic $\mathfrak{sl}_3$-webs [37].

The $\mathfrak{sl}_2$-webs that do not satisfy $\mathrm{sh} = 0$ will be all $\mathfrak{sl}_2$-webs with internal circles (aka closed $\mathfrak{sl}_2$-*sub*-webs) and all $\mathfrak{sl}_3$-webs with internal digons or closed $\mathfrak{sl}_3$-sub-webs.

A bigger example would be the $\mathfrak{sl}_4$-web from the Examples 3.26 and 4.10. Here the resulting 4-multitableaux will be

$$\vec{T}_c = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 14 & & \end{array}, \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 12 & 13 & \\ 17 & & & \end{array}, \begin{array}{ccccc} 1 & 2 & 3 & 4 & 19 \\ 5 & 6 & 11 & & \\ 15 & 16 & 18 & & \end{array}, \begin{array}{ccccc} 1 & 2 & 3 & 4 & 19 \\ 5 & 6 & 9 & 10 & \\ 7 & 8 & 12 & 13 & \end{array} \right)$$

Thus, by the Theorem 4.19 below, this $\mathfrak{sl}_4$-web is not dual-canonical because the degree of $\vec{T}_c$ is -1. In fact, only the node labeled 13 is not of degree zero, but of degree -1.

We are now ready to state the condition for a $\mathfrak{sl}_n$-web to be dual canonical. It is worth noting that the conditions (b) and (c) can be checked by the algorithm from Definition 4.13. Recall the shift $d(\vec{k})$ in the definition of the Kuperberg bracket, see 3.2.13.

Theorem 4.19. *Given a $\mathfrak{sl}_n$-web $u \in W_n(\vec{k})$. The following is equivalent.*

(a) *The $\mathfrak{sl}_n$-web u is a dual canonical basis element.*
(b) *The evaluation of $w = u^*u$ satisfies $\mathrm{ev}(w) = q^{-d(\vec{k})}(1 + \mathrm{rest}(w))$ with $\mathrm{rest}(w) \in q\mathbb{N}[q]$ (positive exponent property).*
(c) *The set ev_u does not contain n-multitableaux $\vec{T}$ with $\deg_{\mathrm{BKW}}(\vec{T}) \leq 0$ except the canonical n-multitableaux $\vec{T}_c$ which is of degree zero.*

Moreover, a $\mathfrak{sl}_n$-web $u \in W_n(\vec{k})$ that does contain a closed $\mathfrak{sl}_n$-sub-web is never dual canonical.

One could hope that Theorem 4.19 is fruitful in both direction, i.e. one can hope to obtain results about *canonical* bases of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight modules because, as pointed out by Mackaay [50] in Corollary 4.21, the notions of canonical and dual canonical switch under q-skew Howe duality.

Proof. Let us first explain why (b)$\Leftrightarrow$(c). The difference is that ev_u contains all flows on u, while ev_{u^*u} contains all possible ways to glue flows on u together. Still (b) and (c) are equivalent: The weight of a flow f on $w = u^*u$ is given by the sum of the weights of two flows f_b and f_t on the bottom and top part respectively. But by Theorem 4.15, Proposition 3.24 and the properties of the q-Shapovalov form $\langle \cdot, \cdot \rangle_{\mathrm{Shap}}$ we see that (b)$\Leftrightarrow$(c).

To be precise, we have

$$\deg_{\mathrm{BKW}}(u_f^*) = \deg_{\mathrm{BKW}}(u_f) - d(\vec{k})$$

by duality. Thus, (c)$\Rightarrow$(b) since, under the assumption that (c) is true, there can be only one flow of degree $-d(\vec{k})$ on u^*, namely the “dual” of the canonical on u.

Furthermore, the existence of a non-canonical flow f on u with degree ≤ 0 gives, again by duality, a non-canonical flow on $w = u^*u$ of degree ≤ 0 even after shifting everything by $d(\vec{k})$. Thus, by Theorem 4.15, (b) can not be true. Moreover, a canonical flow f_c always exists and has degree lower or equal zero by Lemma 4.17. That is, if f_c has negative degree, then, by Theorem 4.15 and duality again, (b) can not be true. Hence, $\neg$(c) $\Rightarrow$ $\neg$(b).

(a)$\Rightarrow$(b): This follows from Theorem 4.15, because the evaluation $\mathrm{ev}(w)$ is (up to a shift) the q-Shapovalov form $\langle u, u \rangle_{\mathrm{Shap}}$. By the discussion above the unique pre-canonical structure is given

by the bar-involution ϕ, the elementary tensors and Lusztig's bilinear form $(\cdot,\cdot)_{\mathrm{Lusz}} = \overline{\langle\cdot,\phi(\cdot)\rangle}_{\mathrm{Shap}}$. Hence, a $\mathfrak{sl}_n$-web u that does not satisfy (b) can not satisfy the positive exponent property.

(b)⇒(a): Recall that we already know that the q-Shapovalov form is the Kuperberg form is the evaluation result from Theorem 4.15. Thus, we only need to check that u is bar-invariant and satisfies Equation 4.1.3 with $\mathrm{sh} = 0$, $c(u, \vec{\lambda}_{T'}) \in q\mathbb{N}[q]$ and a strict inequality for the sum. Then, because a dual canonical structure is unique (if it exists), we can conclude that the $\mathfrak{sl}_n$-web u is dual canonical.

We observe that Lemma 4.9 ensures that u can be written as a sequence of $F_i^{(j)}$'s acting on a highest weight vector. Hence, since $\phi(F_i^{(j)}) = F_i^{(j)}$, the bar-invariance follows.

Moreover, the second condition follows from Lemma 4.17 (because (b)⇔(c)) together with Theorem 3.28. Thus, (b) is a sufficient condition for u to be dual canonical.

If u has a closed $\mathfrak{sl}_n$-sub-web w, then, since this corresponds to a multiplication by the quantum number $\mathrm{ev}(w)$ by Theorem 4.15 and the canonical flow corresponds to a negative degree of $\mathrm{ev}(w)$, the condition (c) can not be satisfied. □

We can now state a very explicit, pictorial condition for a $\mathfrak{sl}_2$-web u to be dual canonical. This condition is well-known in the case $n = 2$.

There is also a pictorial condition for the $\mathfrak{sl}_3$-webs that was recently found by Robert [63]. We could re-prove his theorem using Theorem 4.19. But it is a straightforward and not very short inductive case-by-case check on the total length of the $\mathfrak{sl}_3$-webs. We omit it here to keep the length of the paper in reasonable boundaries (ok, we failed).

To be a little bit more precise, one takes the last ladder away to obtain $u^<$. Then one uses induction and either extend a non-canonical flow of degree ≤ 0 on $u^<$ to u. The crucial case is then the case when u has an fully internal hexagon as in the counterexample already found by Morrison and Nieh [58]. Namely, in this case $u^<$ is dual canonical, but u is not: One can extend a flow on $u^<$ of degree 1 by a ladder of degree -1 to a non-canonical flow on u with degree 0.

The proof in the end turns out to be a lengthy case-by-case check where the crucial step is related to a 3-multitableaux with removable nodes. For $\mathfrak{sl}_n$-webs this proof would not generalize since the number of possibilities for a n-multitableaux to have removable nodes will grow for bigger n.

But the proof will be an almost immediately consequence of Theorem 4.19 in the case $n = 2$.

Proposition 4.20. *A $\mathfrak{sl}_2$-web $u \in W_2(\vec{k})$ is dual canonical iff it contains no internal circles.*

The reason why this case is so easy is that we only have 2-multitableaux and two (important) colors, i.e. one node in the left or right entry.

Proof. That an internal circle in u is a sufficient condition for u to be not dual canonical is already part of Theorem 4.19.

Assume now that u does not have internal circles. Therefore, u consists only of closed arcs. Observe that the canonical flow f_c on u has its arcs labeled $\{1\}$ and is therefore of degree zero. If f is a non-canonical flow on u, then at least one arc is labeled $\{2\}$. One can go stepwise from f_c to f by changing the flows on different arcs from $\{1\}$ to $\{2\}$. Each such step raise the degree by one, since the node for $\{1\}$ is in the right entry of the corresponding 2-multitableaux, while the node for $\{2\}$ is in the left and has therefore an addable node, i.e. the one with the same label in the 2-multitableau for the canonical flow.

Thus, u is dual canonical by Theorem 4.19 part (c). □

4.2. **Connection to the colored $\mathfrak{sl}_n$-link polynomials.**

4.2.1. *Short overview.* In this section we discuss another application of our evaluation algorithm from Theorem 4.15. That is, we show how the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial can be computed using the language of n-multitableaux.

We start by recalling the MOY-calculus from [59] (adopted to our notation) in Subsection 4.2.2. Moreover, recall that a $\mathfrak{sl}_n$-*colored* link diagram L_D is a diagram of a fixed link together with a fixed *color* $a \in \{0, \dots, n\}$ for each component of the diagram. These colors correspond to the a-th fundamental $\mathbf{U}_q(\mathfrak{sl}_n)$-representation $\Lambda^a\bar{\mathbb{Q}}^n$ and hence, they should change to the duals at cups and caps and are otherwise fixed for each strand.

After recalling the MOY-calculus, we show in Subsection 4.2.3 how a colored, oriented link diagram L_D[9] can be translated into a string of $F_i^{(j)}$'s acting on a suitable highest weight vector v_h (depending on L_D) together with a *colored braiding* $T^i_{a,b}$ for $a, b \in \{0, \dots, n\}$, where the $T^i_{a,b}$ are certain sums of strings of $F_i^{(j)}$'s, see Lemma 4.30. Under q-skew Howe duality this gives rise to a sum of certain closed $\mathfrak{sl}_n$-webs w_l with $l = 1, \dots, 2^{\mathrm{cr}}$ (where cr is the number of crossings of the link diagram L_D). These w_l can be evaluated using the Kuperberg bracket $\langle w_l\rangle_{\mathrm{Kup}}$ which can be *explicitly computed* using our algorithm from Theorem 4.15. The colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial $\langle L_D\rangle_n$ of L_D will then be a sum of certain shifts of these quantum numbers, see Theorem 4.31.

It is worthwhile to note that the invariance under the Reidemeister moves can be directly proven in our set-up. We sketch how this works in the proof of Theorem 4.31. To summarize our alternative proof of the invariance: The invariance under the Reidemeister moves is just a *consequence* of the higher quantum Serre relations (which e.g. can be found in Chapter 7 in [49]).

Remark 4.21. Although we do not do it here explicitly, it is not hard to adopt the discussion in this section to tangles. While the result for a link is a quantum number in $\mathbb{Z}[q, q^{-1}]$ (a Laurent polynomial in q with *integer* coefficients), the result for a tangle is a matrix of quantum numbers.

To see this note that the invariant is an intertwiner of $\mathbf{U}_q(\mathfrak{sl}_n)$-representations which we, under q-skew Howe duality, see as a certain string of $F_i^{(j)}$'s acting on a $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight space $W_n(\vec{k}_b)$ at the bottom to another $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight space $W_n(\vec{k}_t)$ at the top. In the case of a link the bottom one will be the highest $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight space and the top the lowest $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight space of the $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight module $W_n(\Lambda)$. Both are of dimension 1. Hence, the whole results is given by a certain quantum number. For a tangle the weight spaces $W_n(\vec{k}_b)$ and $W_n(\vec{k}_t)$ do not have to be one dimensional.

4.2.2. *The MOY-calculus.* We start by recalling the *colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial* $\langle L_D\rangle_n$ of a colored link diagram L_D following the approach of Murakami, Ohtsuki and Yamada from [59], i.e. using the so-called *MOY graph polynomial* $\langle w\rangle_{\mathrm{MOY}}$ of a closed $\mathfrak{sl}_n$-web w. It is worth noting that some authors use the notion of *(embedded) closed, trivalent MOY graph* and use Γ to denote these graphs. Moreover, some authors say *colored edges* instead of *labeled edges* and *states* instead of *flows* and use another notion of weights. But these are the same as the notions of closed $\mathfrak{sl}_n$-webs, labeling of these $\mathfrak{sl}_n$-webs and flows on these $\mathfrak{sl}_n$-webs and shifted version of our notion of weight. So we stick to our conventions and hope that the reader does not

[9] In the following we always denote by L_D an oriented link diagram. If we draw pictures, then we use a different arrow for this orientation to make it easier to distinguish it from the orientation of $\mathfrak{sl}_n$-webs.

get confused. To fix notation, we call a crossing *positive* and a crossing *negative* and the difference of their total numbers $|\,\cdot\,|$ and $|\,\cdot\,|$ the *writhe* $\mathrm{w}(L_D) = |\,\cdot\,| - |\,\cdot\,|$ of the diagram.

Definition 4.22. (**MOY graph polynomial**) Let w be a closed $\mathfrak{sl}_n$-web and let $V(w)$ and $E(w)$ be the sets of its vertices and edges. Let $c\colon E(w) \to \mathbb{N}$ be the function that assigns to edges $e \in E(w)$ its *label* (or color) $c(e) \in \mathbb{N}$. Moreover, for a fixed flow f on w let $\mathrm{f}\colon E(w) \to \mathfrak{P}(\{n, \dots, 0\})$ be the function that assigns to each edges $e \in E(w)$ its *flow* (or state) $\mathrm{f}(e) \in \mathfrak{P}(\{n, \dots, 0\})$

Recall that for each vertex $v \in V(w)$ and a fixed flow w_f the notation $\mathrm{wt}^v(w_f)$ denotes the weight of the vertex v with respect to w_f (see Definition 3.25). Define the ***(total) shifted weight*** $\mathrm{wt}(v, w_f)$ and $\mathrm{wt}^t(v, w_f)$ by

$$\mathrm{wt}(v, w_f) = q^{\frac{\mathrm{c}(e_1)\mathrm{c}(e_2)}{2} - \mathrm{wt}^v(w_f)} \quad \text{and} \quad \mathrm{wt}^t(v, w_f) = \prod_{v \in V(w)} \mathrm{wt}(v, w_f),$$

where $e_1, e_2 \in E(w)$ are the two unique incoming or outgoing edges at v.

Define for a fixed flow f on w a graph by replacing each edge $e \in E(w)$ by $\mathrm{c}(e)$ parallel edges. Then assign to each of these edges a different element of $\mathrm{f}(e)$. Then connect the new edges with the same element of $\mathfrak{P}(\{n, \dots, 0\})$. From this we get a *collection of embedded, oriented, labeled circles* that we denote by $\mathcal{C}$ and we denote the label of each $C \in \mathcal{C}$ by $\mathrm{f}(C)$. Moreover, denote by $\mathrm{rot}(C)$ the *orientation* of the circle C, i.e. $\mathrm{rot}(C) = 1$ if the orientation is counter-clockwise and $\mathrm{rot}(C) = -1$ otherwise. Note that there are some for us unimportant technicalities how to obtain these circles, see [59].

The *rotation number* $\mathrm{rot}(w_f)$ is then defined by

$$\mathrm{rot}(w_f) = \sum_{C \in \mathcal{C}} \mathrm{rot}(C)\mathrm{f}(C).$$

Then the $\mathfrak{sl}_n$*-MOY graph polynomial* of w is defined by

$$\langle w \rangle_{\mathrm{MOY}} = \sum_{f \in Fl(w)} \mathrm{wt}^t(v, w_f) q^{\mathrm{rot}(w_f)} \in \mathbb{N}[q, q^{-1}],$$

where $Fl(w)$ denotes the set of all flow lines on w.

Recall that Murakami, Ohtsuki and Yamada showed in [59] the following (the proof itself is spread over their paper, but most of it is in Section 2 and the appendix).

Theorem 4.23. ***(Murakami, Ohtsuki and Yamada)*** *The MOY graph polynomial $\langle \cdot \rangle_{\mathrm{MOY}}$ satisfies all the relations of the $\boldsymbol{U}_q(\mathfrak{sl}_n)$-spider $\boldsymbol{Sp}(\boldsymbol{U}_q(\mathfrak{sl}_n))$ from Definition 3.12.* $\square$

Moreover, the following seems to be known to the experts. But a proof can for example be found in Wu's paper [79], that is, his Theorem 2.4.

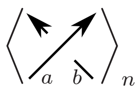
Theorem 4.24. *The MOY graph polynomial $\langle \cdot \rangle_{\mathrm{MOY}}$ is uniquely determined by the relations of the $\boldsymbol{U}_q(\mathfrak{sl}_n)$-spider $\boldsymbol{Sp}(\boldsymbol{U}_q(\mathfrak{sl}_n))$ from Definition 3.12.* $\square$

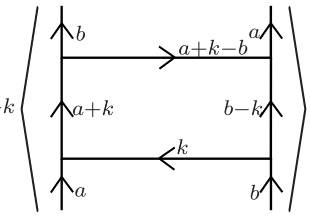
Hence, our notions are the same, something that is not clear from Definition 4.22 above and follows only from the Theorems 4.23 and 4.24. Because of this we use our notation in the following.

Corollary 4.25. *Let $w = v^*u$ be a closed $\mathfrak{sl}_n$-web. Then $\langle w \rangle_{Kup} = \langle w \rangle_{\mathrm{MOY}}$.* $\square$

Recall that Murakami, Ohtsuki and Yamada defined the *colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial* $\langle L_D \rangle_n$ of a colored link diagram L_D in the following way. Note that, using Corollary 4.25, we state everything using the Kuperberg bracket below. Moreover, the reader familiar with [59] should be careful that we use q instead of $q^{\frac{1}{2}}$.

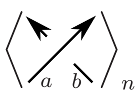
Definition 4.26. **(Colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial)** Let L_D be a colored link diagram. Then the *colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial* $\langle L_D \rangle_n$ of L_D is defined by applying the following to all crossings of L_D. We use

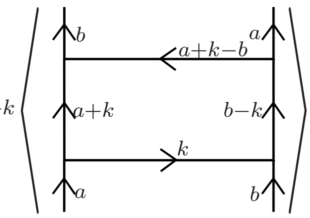


$= \sum_{k=0}^{b} (-1)^{k+(a+1)b} q^{-b+k}$


$_{\mathrm{Kup}}$,

if $b \leq a$, and for $a < b$ we use

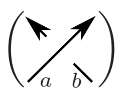


$= \sum_{k=0}^{a} (-1)^{k+(b+1)a} q^{-a+k}$


$_{\mathrm{Kup}}$

for a positive  $_{a,b}$ and almost the same for a negative  $_{a,b}$ with the same colors a, b, but the powers of q above are minus the ones for the positive  $_{a,b}$.

Moreover, for each positive crossing  $_{a,b}$ we introduce the *shift*

s 

$$= \begin{cases} (-1)^{b+1} q^{b(n+1-b)}, & \text{if } a = b, \\ 1, & \text{else,} \end{cases}$$

and the same again up to a multiplication with -1 in the exponent of q for a negative crossing with the same colors.

The *normalized*, colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial of L_D is then defined by

$$\mathrm{RT}_n(L_D) = \langle L_D \rangle_n \cdot \prod_{c_{a,b}} s(c_{a,b}), \tag{4.2.1}$$

where the product runs over all colored crossings. It is worth noting that the total shift in the "uncolored" case $a = b = 1$ is $q^{n \cdot \mathrm{w}(L_D)}$.

The following is Theorem 5.1 in [59]. Note that we sketch an alternative proof later in the proof of Theorem 4.31.

Theorem 4.27. ***(Murakami, Ohtsuki and Yamada)*** *The colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial $\langle \cdot \rangle_n \in \mathbb{Z}[q, q^{-1}]$ is invariant under the second and third Reidemeister moves.*

The normalized, colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial $\mathrm{RT}_n(\cdot) \in \mathbb{Z}[q, q^{-1}]$ is an invariant of links. $\square$

Note that already $\langle \cdot \rangle_n$ is invariant under the Reidemeister moves up to a normalization, i.e. it gives an invariant of framed links. We ignore the normalization in the following.

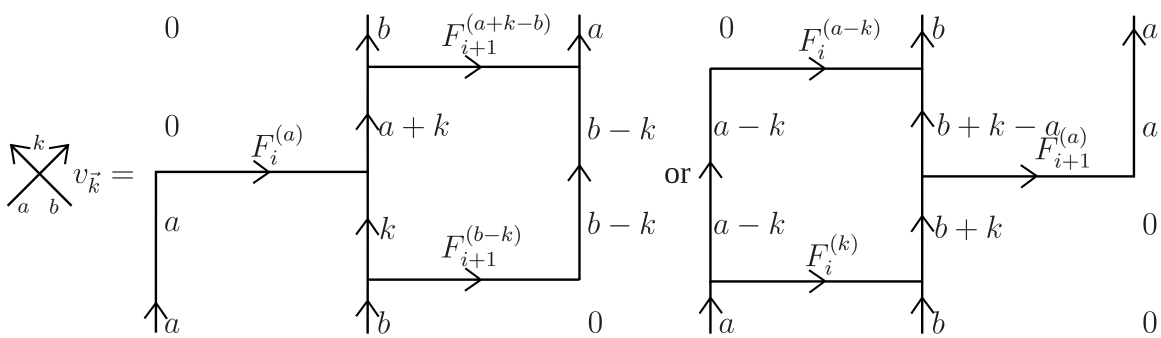
4.2.3. *Our set-up.* The rest of the section is intended to explain how our approach can be used to *calculate* $\langle L_D\rangle_n$ for all colorings using the language of n-multitableaux. Thus, we have explain how a colored link diagram L_D can be translated to our framework using q-skew Howe duality and actions of $F_i^{(j)}$'s on some highest weight vector $v_{(n^\ell)}$. We start by defining the *colored braiding* operators. Recall that we assume that Λ denotes n-times the ℓ-th fundamental $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight and that $W_n(\Lambda)$ denotes the irreducible $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-representation of highest weight Λ. Here we use Equation 3.2.8 again to convert a $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$-weight to a $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight. Recall that we use the notation $\vec{k}$ for such weights.

Definition 4.28. For $a,b\in\{0,\dots,n\}$ let $\vec{k}=(\dots,a,b,0,\dots)$ and $\vec{k}'=(\dots,0,a,b,\dots)\in\mathbb{N}^m$ be $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weights where a is the i-th entry of $\vec{k}$ and the $i+1$-th entry of $\vec{k}'$.

For all $k=0,\dots,\min(a,b)$ the *k-th colored braiding operator* $T^k_{a,b,i}$ acts on the $\vec{k}$-weight space $W_n(\vec{k})$ of $W_n(\Lambda)$ by

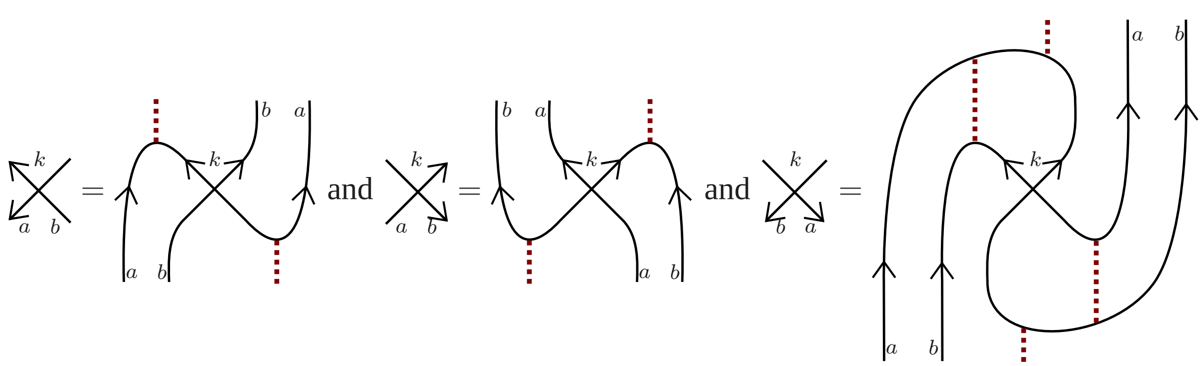
$$T^k_{a,b,i}\colon W_n(\vec{k})\to W_n(\vec{k}'),\, v_{\vec{k}}\mapsto\begin{cases}F_{i+1}^{(a+k-b)}F_i^{(a)}F_{i+1}^{(b-k)}v_{\vec{k}}, & \text{if } b\leq a,\\ F_i^{(a-k)}F_{i+1}^{(a)}F_i^{(k)}v_{\vec{k}}, & \text{if } a<b,\end{cases}$$

for $v_{\vec{k}}\in W_n(\vec{k})$ or in pictures with $T^k_{a,b,i}=$



Note that, if the weights have values <0 or $>n$, then the corresponding diagram is zero due to our convention. The same is true for the action, since it factors through $\Lambda^{<0}\bar{\mathbb{Q}}^n$ or $\Lambda^{>n}\bar{\mathbb{Q}}^n$.

We define the *left* ${}_lT^k_{a,b,i}=$, *right* ${}_rT^k_{a,b,i}=$ and *downwards* ${}_dT^k_{a,b,i}=$ versions by



It is worth noting that this is the "standard" way to define left, right and downwards versions in categories with duals and suitable bi-adjoint caps and cups. Moreover, since we already know that the n-multitableaux framework is isotopy invariant by the discussion in Section 4.1, we could

also define left, right and downwards versions of the braiding operators directly. But this is only important if one cares about *calculation efficiency*, since one could keep the m smaller this way.

The reader is invited to verify that these three definitions correspond to

$$ {}_lT^k_{a,b,i}v_{\vec{k}_l}=F^{(a)}_{i+1}F^{(a)}_iT^k_{b,n-a,i+1}F^{(a)}_{i+3}F^{(a)}_{i+2}v_{\vec{k}_l}\quad\text{and}\quad {}_rT^k_{a,b,i}=F^{(n-b)}_{i+1}F^{(b)}_{i-2}T^k_{n-b,a,i-1}F^{(n-b)}_{i-2}F^{(b)}_{i+1}v_{\vec{k}_r} $$

with the new weights $\vec{k}_l=(\dots,a,b,n,0,0,\dots)$ and $\vec{k}_r=(\dots,n,0,a,b,0,\dots)$ and

$$ {}_dT^k_{a,b,i}v_{\vec{k}_l}=F^{(a)}_{i+2}F^{(a)}_{i+3}F^{(a)}_{i+1}F^{(b)}_{i+2}F^{(a)}_iF^{(b)}_{i+1}T^k_{n-a,n-b,i+2}F^{(b)}_{i+4}F^{(b)}_{i+3}F^{(a)}_{i+2}F^{(a)}_{i+5}F^{(a)}_{i+4}F^{(a)}_{i+3}v_{\vec{k}_d} $$

with $\vec{k}_d=(\dots,a,b,n,n,0,0,0,\dots)$ with a always in the i-th position and the $v_{\vec{k}}$'s are all vectors in the corresponding weight modules for the three $\vec{k}$'s.

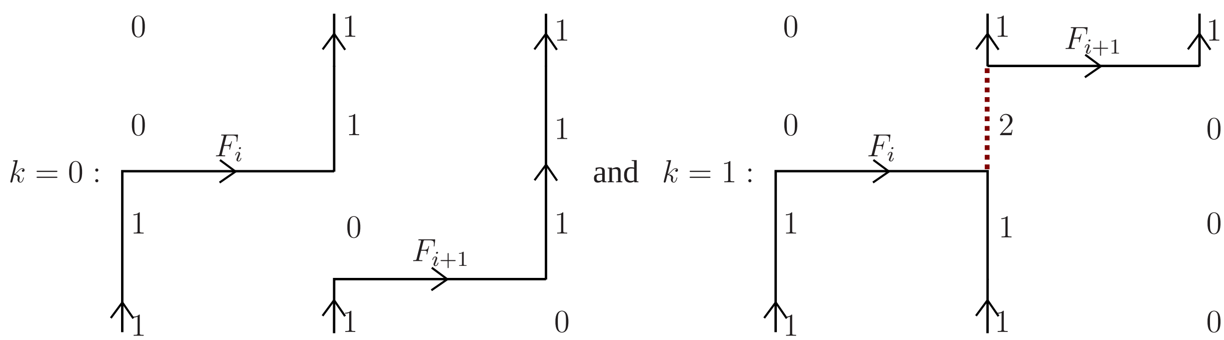
The *positive full braiding operator* $T_{a,b,+i}$ is then defined to be the q-weighted sum

$$ T_{a,b,+i}=\begin{cases}\sum_{k=0}^{b}(-1)^{k+(a+1)b}q^{-b+k}T^k_{a,b,i}, & \text{if } b\le a,\\ \sum_{k=0}^{a}(-1)^{k+(b+1)a}q^{-a+k}T^k_{a,b,i}, & \text{if } a<b.\end{cases} \tag{4.2.2} $$

Moreover, the *negative full braiding operator* $T_{a,b,-i}$ is defined similar but with all powers of q multiplied by the factor -1.

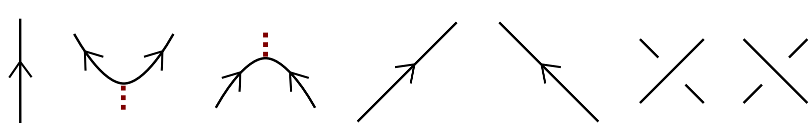
Example 4.29. The Definition 4.28 is a conversion of Definition 4.22 to our framework. The main point is that we have to wiggle the braid a little bit around to write it as a sequence of $F^{(j)}_i$'s.

Let us consider a small $\mathfrak{sl}_2$ example. Let $a=b=1$ and therefore $k=0$ or $k=1$. Then we have essentially two pictures.



These are exactly the two terms in the Kauffman calculus for the Jones polynomial. It is worth noting that in this case (and in fact in all cases with only color 1) the difference between the two strings of $F^{(j)}_i$'s can be seen as an action of the symmetric group S_{m-1}, since we permute F_iF_{i+1} to $F_{i+1}F_i$ (or vice versa).

Let T_D denote a colored, oriented diagram of a tangle. We assume that T_D is in a *general Morse position*. With this we mean that strands of T_D are locally either *identities*, *cups*, *caps*, *shifts*, *overcrossings* or *undercrossings* (with all possible orientations) as illustrated below.



Our approach for calculation is to use the evaluation algorithm. Hence, we will need the following lemma. It is worth noting that implicitly in the proof of the lemma (which is based on the proof of Lemma 4.9) is an algorithm to write a given T_D under q-skew Howe duality.

Lemma 4.30. *Any colored, oriented tangle diagram T_D can be written, using q-skew Howe duality with acting pair $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, as*

$$T_D = \prod_{k=1}^{s} \tilde{F}_{i_k}^{(j_k)} v_{(n^\ell)}, \quad \tilde{F}_{i_k}^{(j_k)} = \begin{cases} F_{i_k}^{(j_k)}, & \textit{for some } i_k \in \{1, \dots, m-1\}, j_k \in \{0, \dots, n\}, \\ T_{a_k, b_k, \pm i_k}, & \textit{for some } a_k, b_k \in \{0, \dots, n\}, i_k \in \{1, \dots, m-1\}, \end{cases}$$

for some $s \in \mathbb{N}$, some highest weight vector $v_{(n^\ell)}$ and marked braiding operators $T_{a_k,b_k,\pm i_k}$ (where the signs should indicate if the corresponding crossing is positive or negative).

Hence, each such tangle diagram T_D can be realized as

$$T_D = \prod_{k=1}^{s} \tilde{F}_{i_k}^{(j_k)} v_{(n^\ell)} = \sum_{j=1}^{t} (-1)^{\mathrm{sgn}_j} q^{\mathrm{sh}_j} \prod_{k_j=1}^{s_j} F_{i_{k_j}}^{(j_{k_j})} v_{(n^\ell)} = \sum_{j=1}^{t} (-1)^{\mathrm{sgn}_j} q^{\mathrm{sh}_j} u_j$$

where sgn_j and sh_j are some constants and all summands are of the same total length $\sum j_{k_j}$. The u_j are certain $\mathfrak{sl}_n$-webs. Moreover, if T_D is a link diagram, then the u_j are all closed $\mathfrak{sl}_n$-webs.

Proof. The proof works very similar to the one given in Lemma 4.9. The reason for this is easily explained, i.e. all the allowed local pieces except the positive and negative crossings are already included in the allowed pieces in Lemma 4.9, where the conversion to ladder moves is as before.

The positive and negative crossings can then be realized as the braiding operators $T_{a,b,i}$ given in Definition 4.28. The signs then just indicate if the crossing is positive or negative.

Thus, all the statements are easy to verify following Lemma 4.9 and we leave the details to the reader. □

Using the last part of Lemma 4.30 we can therefore define the *evaluation* $\mathrm{ev}(L_D)$ of a colored, oriented link diagram L_D to be

$$\mathrm{ev}(L_D) = \sum_{j=1}^{t} (-1)^{\mathrm{sgn}_j} q^{\mathrm{sh}_j} \mathrm{ev}(w_j),$$

where $\mathrm{ev}(w_j)$ denotes our evaluation algorithm from Definition 4.13.

Theorem 4.31. *Let L_D be a colored, oriented link diagram. The evaluation $\mathrm{ev}(L_D)$ is invariant under the second and third Reidemeister moves and isotopies. Moreover,*

$$\mathrm{ev}(L_D) = \langle L_D \rangle_n,$$

i.e. the evaluation algorithm gives the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial. The normalized colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial can be obtained by a shift.

Proof. This is only an assembling of pieces now. In fact, it follows from Theorem 4.15 together with the Corollary 4.25.

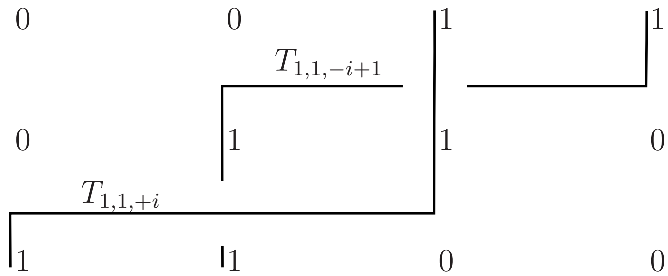
But there is an alternative way to prove the statement in our set-up as we sketch here. Because of Theorem 4.15 we note that we already have the isotopy invariance in our set-up. Thus, it suffices to restrict to braids (the braid is oriented upwards). We sketch how to show the invariance for the second Reidemeister move by restricting to the "uncolored" case $a = b = 1$. It will be a consequence of the Serre relations from Definition 3.14. The same is true for the "uncolored" third Reidemeister move as we invite the reader to check. The invariance in the "honestly colored" case

follows in the same vein using the higher Serre relations that can be for example found in Chapter 7 of Lusztig's book [49].

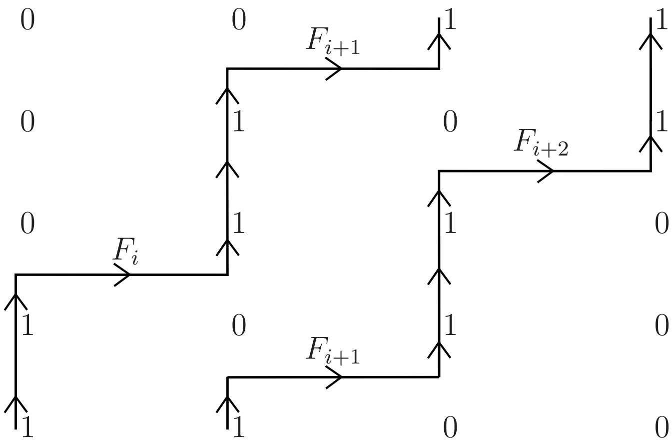
The invariance under the second Reidemeister move in our case can be proven by checking that

(4.2.3) $\quad T_{1,1,\mp i+1}T_{1,1,\pm i}v_{\dots,1,1,0,0,\dots} = F_{i+1}F_{i+2}F_iF_{i+1}v_{\dots,1,1,0,0,\dots},$ with the first 1 in the i-th entry.

Or in pictures (the other possibility can be proven analogously): The move



has to be



We only do the algebra now and encourage the reader to draw the pictures. Factoring the left side of Equation 4.2.3 using the definition from 4.2.2 gives the term (we use $v = v_{\dots,1,1,0,0,\dots}$)

$$(F_{i+1}F_{i+2}F_iF_{i+1} - q^{+1}\cdot F_{i+1}F_{i+2}F_{i+1}F_i - q^{-1}\cdot F_{i+2}F_{i+1}F_iF_{i+1} + F_{i+2}F_{i+1}F_{i+1}F_i)v.$$

Therefore, it suffices to show that

$$F_{i+2}F_{i+1}F_{i+1}F_iv \overset{!}{=} q^{+1}\cdot F_{i+1}F_{i+2}F_{i+1}F_iv + q^{-1}\cdot F_{i+2}F_{i+1}F_iF_{i+1}v.$$

Since $F_{i+1}^2v = 0$, we see that

$$q^{-1}\cdot F_{i+2}F_{i+1}F_iF_{i+1}v = \frac{q^{-1}}{[2]}\cdot F_{i+2}F_{i+1}^2F_iv$$

by using the Serre relations on the right three F's. Using the Serre relations now to the three left F's of the other term gives

$$\frac{q^{+1}}{[2]}\cdot F_{i+2}F_{i+1}^2F_iv + \frac{q^{-1}}{[2]}\cdot F_{i+2}F_{i+1}^2F_iv = F_{i+2}F_{i+1}^2F_iv = F_{i+2}F_{i+1}F_{i+1}F_iv.$$

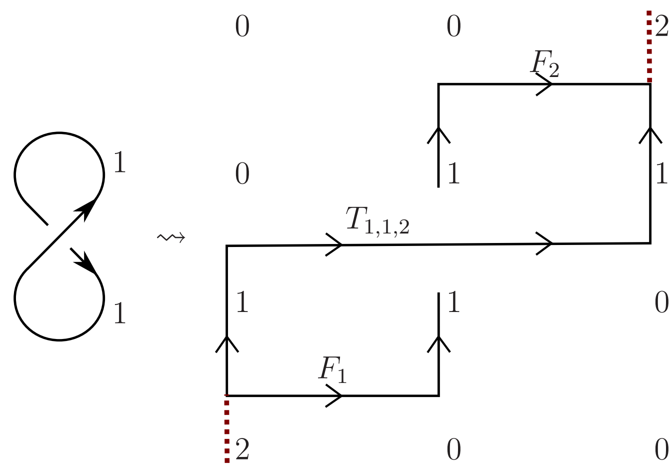
Thus, we obtain the invariance. The other cases follow similar. □

It is worth noting that Theorem 4.31 can also be used to calculate the tangle invariant intertwiner related to the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-tangle invariant. Moreover, this gives an algorithm to calculate the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial or tangle invariant, since all steps are given by an algorithm.

4.2.4. *Two examples.* We conclude this section by giving two hopefully illustrating examples.

It should be noted, since empty shifts do not change anything interesting, we sometimes do not use them in the following, e.g. in order to go from the highest to the lowest weight one would have to do empty shifts at the end to order all non-zero entries to the right.

Example 4.32. Let us consider the following small, but illustrating example how to realize a certain diagram of the unknot U_D as such a sum of $F_i^{(j)}$'s. Here we use $n=2$ and strands are only colored with color 1. Note that this example belongs to Example 4.14.



Hence, we can write the unknot as (beware that it has an undercrossing)

$$U_D = F_2 T_{1,1,2} F_1 v_{(2^1)} = qF_2F_2F_1F_1v_{(2^1)} - F_2F_1F_2F_1v_{(2^1)}.$$

We should note that we are cheating a little bit here, since, if we would strictly follow the algorithm, then we would have to re-write the right pointing crossing as in Definition 4.28 and we would get

$$\begin{aligned} U_D &= F_4F_2F_1T_{1,1,2}F_1F_4F_3F_2^{(2)}v_{(2^2)} \\ &= qF_4F_2F_1F_2F_3F_1F_4F_3F_2^{(2)}v_{(2^2)} - F_4F_2F_1F_3F_2F_1F_4F_3F_2^{(2)}v_{(2^2)} \end{aligned}$$

as we invite the reader to check. But the power of Theorem 4.15, which is based on Proposition 4.12, i.e. the n-multitableaux calculus is isotopy invariant, leaves us room to speed the algorithm up.

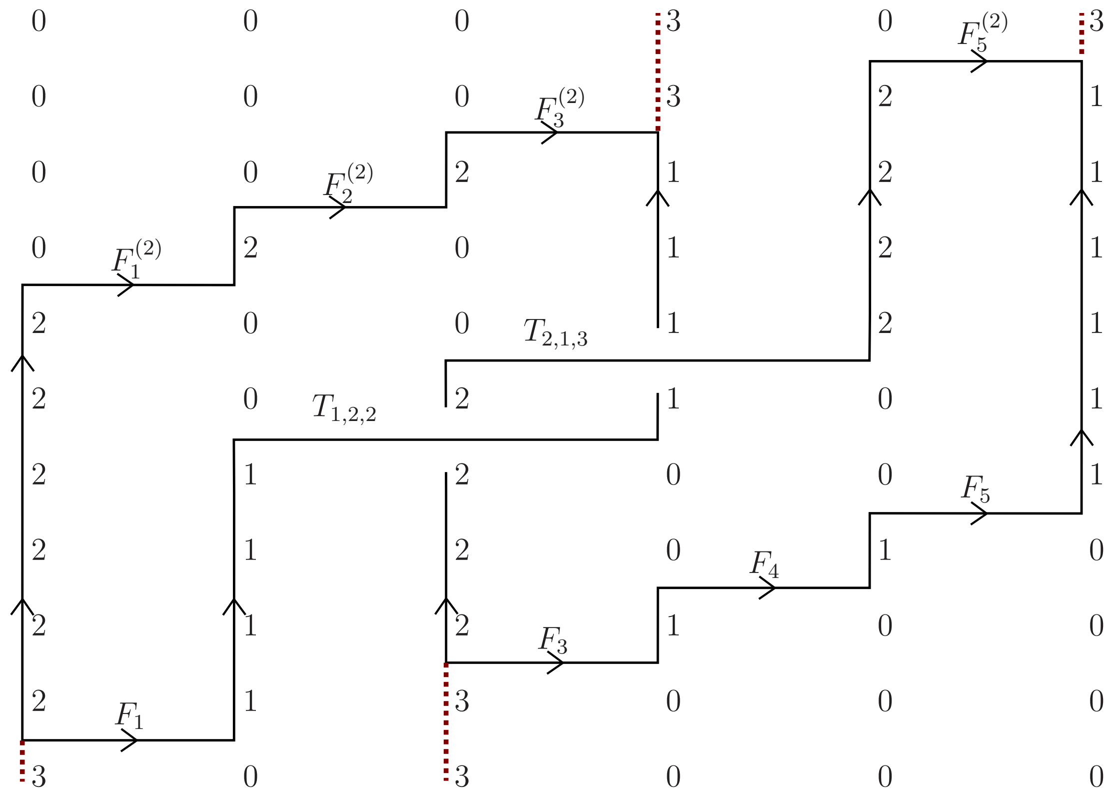
Hence, as we have already calculated in Example 4.14 before, the left summand gives four 2-multitableaux of degrees $2, 0, -2$ and the right summand two of degrees $1, -1$. Thus,

$$\mathrm{ev}(U_D) = q(q^2+2+q^{-2}) - (q+q^{-1}) = q^3+q = q^2[2],$$

which is, up to a normalization, the polynomial $[2]$ of the trivial diagram. The normalization factor given in Definition 4.26 is indeed $q^{-2} = q^{2\mathrm{w}(U_D)}$ in this case.

Note that, although we allow any Morse position of a colored link diagram, it could be sometimes better to restrict to closures of braids (as in Example 4.33 - beware that the one we show there is not the trace closure). With the presentation above it is for example not possible to have a $a=b=1$ colored crossing for $n>2$. But this would be possible for any closure of a braid.

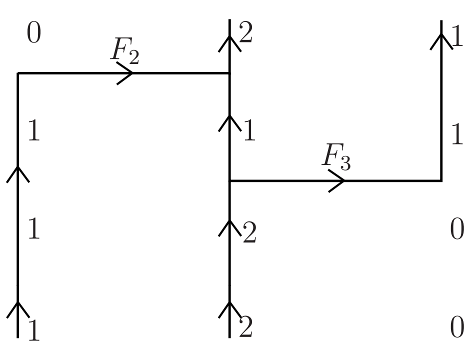
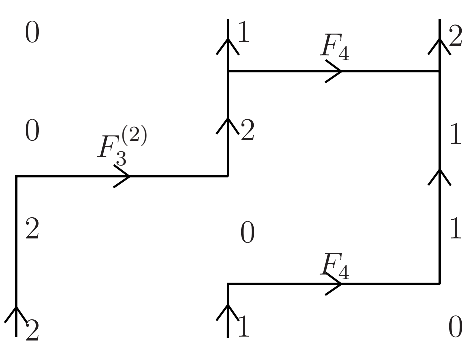
Example 4.33. A more demanding, but also more interesting, example is the Hopf link given below. Our space here is limited, so we only sketch the calculation. But the reader is encouraged to do the full calculation: We claim that doing so gives a full feeling how bigger cases work.



In this case we want to calculate the colored $\mathfrak{sl}_3$-link polynomial using $\dot{\mathbf{U}}_q(\mathfrak{sl}_6)$-weight representations, i.e. $n = 3$ and $m = 6$. Moreover, the colors are illustrated above, that is we have the two braiding operators $T_{1,2,2}$ (bottom) and $T_{2,1,3}$ (top). Thus, we choose the orientations of the Hopf link to point upwards, i.e. both crossings should be ⤧. Both of them correspond to two summands. Thus, we have four summands in total. Moreover, we see that (in the picture above we skipped the empty shift $F_2^{(3)}$ at the bottom and we can totally ignore the empty shift at the top)

$$\mathrm{Hopf} = F_5^{(2)} F_3^{(2)} F_2^{(2)} F_1^{(2)} T_{2,1,3} T_{1,2,2} F_5 F_4 F_3 F_1 F_2^{(3)} v_{(3^2)}.$$

The first operator gives the summands $-q^{-1}F_2F_3\mathbb{1}$ and $\mathbb{1}F_3F_2$ and the top one gives $-q^{-1}F_4F_3^{(2)}F_4$ and $F_4^{(2)}F_3^{(2)}\mathbb{1}$. Recall from Definition 4.28 that the braiding operators will have three terms $F_i^{(j)}$'s and we have indicated the trivial one by $\mathbb{1}$. Or in local pictures

$F_2F_3\mathbb{1}$:  and $F_4F_3^{(2)}F_4$: 

The reader should check that the other two look like the right case in Example 4.29 with different numbers. We now have to follow the algorithm to generate for each of the four possibilities the sets of 3-multitableaux. Note that the string of $F_i^{(j)}$'s before the first braiding operator, denoted by F_b, "opens" two components that will eventually connect later. Both correspond to three possible flows and the evaluation algorithm will generate nine 3-multitableaux. They will all look like

$$\left(\begin{ytableau} 1 & - & - & - \\ \cdot \end{ytableau}\,,\ \begin{ytableau} 1 & - & - & - \\ \cdot \end{ytableau}\,,\ \begin{ytableau} 1 & - & - & - \\ \cdot \end{ytableau}\right) \qquad \left(\begin{ytableau} 1 & 3 & 4 & 5 \end{ytableau}\,,\ \begin{ytableau} 1 \end{ytableau}\,,\ \begin{ytableau} 1 \\ 2 \end{ytableau}\right),$$

where the number 2 is allowed to appear in the node marked $\cdot$ and the numbers $3, 4$ and 5 (in order) are allowed to appear in the nodes marked $-$. An explicit example is illustrated above.

The four possibilities how the two braiding operators can be composed will kill some of them and create new ones while we follow the evaluation algorithm. For example, the evaluation of $F_2F_3F_bv_{(3^2)}$ will raise this number to twelve 3-multitableaux because the F_3 can be place in two different positions for each of the nine 3-multitableaux. But the F_2 will kill some of them, since to place a node of residue 2 is only possible if we see a hook. For example, the left of the possible two extensions of the upper right example does not have such a hook, while the right one has.

$$\left(\begin{ytableau} 1 & 3 & 4 & 5 \end{ytableau}\,,\ \begin{ytableau} 1 & 6 \end{ytableau}\,,\ \begin{ytableau} 1 \\ 2 \end{ytableau}\right) \qquad \left(\begin{ytableau} 1 & 3 & 4 & 5 \end{ytableau}\,,\ \begin{ytableau} 1 \end{ytableau}\,,\ \begin{ytableau} 1 & 6 \\ 2 & 7 \end{ytableau}\right).$$

If we extend the string now by $F_4^{(2)}F_3^{(2)}$, then we see that the first will kill most of the possibilities. For example it is not possible to add two nodes of residue 3 to the right 3-multitableaux above. This is due to the fact that $F_3^{(2)}$ corresponds to a cap. The $F_4^{(2)}$, which corresponds to a cup, will then create new possibilities. Following this process to the end and calculate the degrees we see that we will get to

$$\langle \mathrm{Hopf}\rangle_3 = q^{-2}[2]^2[3] - 2q^{-1}[2][3] + [3]^2,$$

which is the corresponding colored quantum polynomial. It is worth noting that the empty shift at the end (we skipped it above) does not change anything.

Remark 4.34. Different colors for the components are easy to implement if one already has one way to write a link diagram as a string of $F_i^{(j)}$'s. One only has to change certain divided powers of some of the $F_i^{(j)}$'s. For example, if we want to consider two crossings with both strings colored 1 for the Hopf link diagram from Example 4.33, then we only have to change the "right" part of the generating string. To be precise, we can generate the same diagram with these other colors by

$$\mathrm{Hopf} = F_5F_3^{(2)}F_2^{(2)}F_1^{(2)}T_{1,1,3}T_{1,1,2}F_5^{(2)}F_4^{(2)}F_3^{(2)}F_1F_2^{(3)}v_{(3^2)}.$$

Moreover, changing to bigger n is also easy. Again one only has to change certain divided powers. For example,

$$\mathrm{Hopf} = F_5^{(4)}F_3^{(4)}F_2^{(4)}F_1^{(4)}T_{4,1,3}T_{1,4,2}F_5F_4F_3F_1F_2^{(5)}v_{(5^2)}$$

gives the Hopf link diagram with crossings colored with 1 and 4. From this we can again produce all the other colorings. Note that for big n the whole evaluation of $\mathfrak{sl}_n$-webs will more demanding than for small n, since we have to use n-multitableaux to calculate the evaluation, but we still use only $\dot{\mathbf{U}}(\mathfrak{sl}_m)$-highest weight theory. To summarize, for each link diagram L_D there is a m such that $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory governs the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial for *all* n and *all* possible colors.

Thus, for each link L there is a minimal such m. A question that seems to be important for that is related to the braid index of the link L, i.e. the minimal number of needed strands to present it as a braid, and the number of crossings of such a presentation, compare to the figures above. Moreover, in the Examples 4.32 and 4.33 we just presented the two links in some way. One could also do it directly as a trace (or classical) closure of a braid. To see this note that one can use the local shifts as indicated in the proof of Lemma 4.9.

Remark 4.35. In fact we think that our approach can be made much more efficient. This would be interesting if one would want to write a computer program for calculations. We think this would be worthwhile since the n-multitableaux language is combinatorial and easy to understand for a computer. But this has not yet been done.

5. ITS CATEGORIFICATION

5.1. **A cellular basis for matrix factorizations.**

5.1.1. *Short overview.* In this section we are going to explain how the extended growth algorithm for $\mathfrak{sl}_n$-webs can be used as a *growth algorithm* for homomorphisms of the associated matrix factorizations. This algorithm gives a basis for the space of such homomorphisms $f\colon \widehat{u} \to \widehat{v}$ between *any* two $\mathfrak{sl}_n$-webs u and v with the same $\vec{k}$ at the boundary as we show in Theorem 5.14.

Our construction is motivated by the Hu and Mathas basis for the cyclotomic KL-R algebra. In fact, it can be thought as a $\mathfrak{sl}_n$-web version of the HM-basis: We show, by using Theorem 5.14 and the combinatorial identification from Section 4.1, in Theorem 5.16 that the $\mathfrak{sl}_n$-web algebra $H_n(\vec{k})$ is *graded isomorphic* to a certain idempotent truncation of the thick cyclotomic KL-R algebra $\check{R}_\Lambda$. Thus, our basis can be seen as a "thick version" of the HM-basis. With this we are able to extend the categorified q-skew Howe duality from Theorem 3.35 to the thick case.

Moreover, our basis turns out to be a *graded cellular basis* of the $\mathfrak{sl}_n$-web algebra $H_n(\vec{k})$. The reason for this is that the basis we describe has three main advantages, namely it is very *general*, given *locally* using an *inductive* process and can be stated *purely combinatorial* using the language of n-multitableaux.

The first property allows us to define it for literally any $\mathfrak{sl}_n$-webs u and v - something that will turn out to be very useful for the *computation* of Khovanov-Rozansky $\mathfrak{sl}_n$-homologies later in Section 5.2. The second and the third property give us a method for calculating the multiplication similar to the framework of Hecke algebras - usually very demanding, but at least possible.

We note that Theorem 5.16 can be seen as a categorification of Lemma 4.9 in the sense that *all* relations in the $\mathfrak{sl}_n$-matrix factorization (or alternatively $\mathfrak{sl}_n$-foam) framework we use follow from the ones in the thick cyclotomic KL-R algebra.

It is worth noting that the growth algorithm we are going to define is very similar to the one the author has worked out for $\mathfrak{sl}_3$-foams in [75]. In fact, everything can also be stated using $\mathfrak{sl}_n$-foams in the sense of Queffelec and Rose [61].

5.1.2. *The dotted idempotent.* We start and give the definition of the idempotent for $\vec{\lambda}$, denoted by $e(\vec{\lambda})$. Recall that we choose and fix n and ℓ and that there is a constant $c(\vec{k})$ that only depends on the weight $\vec{k}$. Note that, since $\vec{\lambda}$ corresponds to a state string $\vec{S}$ which includes the $\vec{k}$, the $\vec{\lambda}$ determines $c(\vec{k})$.

Definition 5.1. **(Idempotent associated to** $\vec{\lambda}$) Given a n-multipartition $\vec{\lambda}$ with $c(\vec{k})$ nodes filled with non-repeating $k \in \{1, \dots, c(\vec{k})\}$, we can associate to it a certain *idempotent*, denoted by $e(\vec{\lambda})$, using the following rules. Define a sequence of F_k's for $\vec{\lambda}$ by (with $r(\vec{\lambda})$ as in Definition 3.10)

$$\mathrm{qH}(\vec{\lambda}) = \prod_{k=1}^{c(\vec{k})} F_{r(\vec{\lambda})_k} = F_{r(\vec{\lambda})_{c(\vec{k})}} \cdots F_{r(\vec{\lambda})_1} \quad \text{with } r(\lambda) = (r(\lambda)_1, \dots, r(\lambda)_{c(\vec{k})}).$$

Define a $\mathfrak{sl}_n$-web $u_{\vec{\lambda}}$ to be the $\mathfrak{sl}_n$-web generated by applying $\mathrm{qH}(\vec{\lambda})$ to a highest weight vector $v_{(n^\ell)}$ (here ℓ is as in Definition 3.1) and use q-skew Howe duality. Then

$$e(\vec{\lambda}) = \mathrm{id}\colon \widehat{u}_{\vec{\lambda}} \to \widehat{u}_{\vec{\lambda}},$$

that is, the identity between the matrix factorization $\widehat{u}_{\vec{\lambda}}$ associated to $u_{\vec{\lambda}}$.

Definition 5.2. **("Dot placement" associated to** $\vec{\lambda}$) Given a n-multipartition $\vec{\lambda}$ as in Definition 5.1 together with its associated idempotent $e(\vec{\lambda})$. Denote by $m(k) = \mathsf{A}^{r \succ N}(T_{\vec{\lambda}^k})$ the number of *addable nodes* after the node N with entry k in $T_{\vec{\lambda}^k}$ with the same residue r as the node N. We denote the "dotted" idempotent associated to $\vec{\lambda}$ by $e(\vec{\lambda})d(\vec{\lambda}) = e(\vec{\lambda}) \circ d(\vec{\lambda})\colon \widehat{u}_{\vec{\lambda}} \to \widehat{u}_{\vec{\lambda}}$, where

$$d(\vec{\lambda}) = \widehat{t}_{c(\vec{k})}^{\,m(c(\vec{k}))} \circ \cdots \circ \widehat{t}_1^{\,m(1)}\colon \widehat{u}_{\vec{\lambda}} \to \widehat{u}_{\vec{\lambda}}.$$

We call it, by abuse of language, *the dotted idempotent associated to* $\vec{\lambda}$.

Lemma 5.3. *The dotted idempotent* $e(\vec{\lambda})d(\vec{\lambda})$ *is always well-defined, that is, it is not zero, and an idempotent iff* $d(\vec{\lambda}) = \mathrm{id}$ *and nilpotent otherwise. For all* n*-multipartitions* $\vec{\lambda}, \vec{\mu}$ *we have*

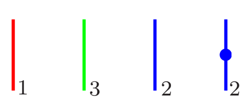
$$e(\vec{\lambda})e(\vec{\mu}) = e(\vec{\mu})e(\vec{\lambda}) = \delta_{\vec{\lambda},\vec{\mu}} e(\vec{\lambda}) = \delta_{\vec{\lambda},\vec{\mu}} e(\vec{\mu}), \text{ with } \delta_{\vec{\lambda},\vec{\mu}} = \begin{cases} 1, & \text{if } r(\vec{\lambda}) = r(\vec{\mu}), \\ 0, & \text{otherwise}. \end{cases}$$

Moreover, we have

$$e(\vec{\lambda}) \circ d(\vec{\lambda}) = d(\vec{\lambda}) \circ e(\vec{\lambda}) \quad \text{and} \quad d(\vec{\lambda}) \circ d(\vec{\mu}) = d(\vec{\mu}) \circ d(\vec{\lambda}),$$

that is, the dotted idempotents for $\vec{\lambda}$ *and* $\vec{\mu}$ *commute.*

Proof. To see that $e(\vec{\lambda})d(\vec{\lambda})$ is well-defined we need two ingredients. The first ingredient is that we have to make the equivalence $\tilde{\Gamma}$ from Equation 3.3.5 explicit. That is, we are going to argue that $e(\vec{\lambda})d(\vec{\lambda})$ is the image of a certain cyclotomic KL-R diagram under $\tilde{\Gamma}$ as illustrated below (the numbers i and colors should illustrate the corresponding $\mathcal{F}_i$'s).

$\tilde{\Gamma}\colon$  $\mapsto e\left(\left(\boxed{1}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}\right)\right) d\left(\left(\boxed{1}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}\right)\right),$

where the residue sequence of $r(\vec{\lambda})$ is (recall our shift) given by $(2, 2, 1, 3)$ and only the node with entry 1 has an addable node (the node with entry 2).

To see that everything works out fine we need our second ingredient, namely the Hu and Mathas basis from [30]. More explicitly, we use their definition of the "dotted" idempotent given in Definition 4.9 and Definition 4.15 in [30]. We denote their diagram associated to $\vec{\lambda}$, by abuse of notation, also by $e(\vec{\lambda})d(\vec{\lambda})$.

We consider now the lift of $e(\vec{\lambda})d(\vec{\lambda})$ to the KL-R algebra, i.e. without taking the cyclotomic quotient (and again use the same notation). Then, by comparing their definition to Definition 5.2, we see that

$$\Gamma_{m,n\ell,n}\colon e(\vec{\lambda})d(\vec{\lambda}) \mapsto e(\vec{\lambda})d(\vec{\lambda}),$$

since the action from 3.3.4 is given explicitly: It sends a dot to a "dot" $\widehat{t}$ and an idempotent as above is send to the identity between the $\mathfrak{sl}_n$-web that can be read of from $r(\vec{\lambda})$.

To see that it is also the image under the cyclotomic equivalence we note that the definition of $\tilde{\Gamma}$ comes from the equivalence given by Rouquier in Proposition 5.6 of his paper [65] which is stated before his Lemma 5.4. Comparing his definition (beware that he uses the lowest weight notation) with our conventions stated throughout the paper shows that

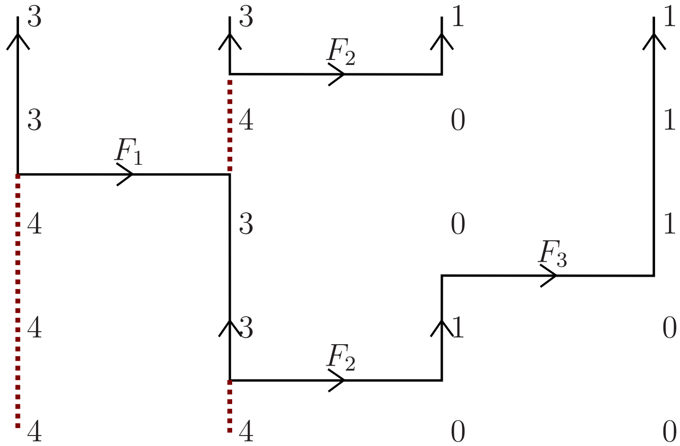
$$\tilde{\Gamma}\colon e(\vec{\lambda})d(\vec{\lambda}) \mapsto e(\vec{\lambda})d(\vec{\lambda}).$$

Thus, applying Corollary 4.16 in [30], we see that the dotted idempotent is well-defined and not zero since $\tilde{\Gamma}$ is faithful. The other statements follow now directly from the corresponding ones in the cyclotomic KL-R setting using the equivalence from 3.3.5. $\square$

Example 5.4. To give an explicit example assume that $n = 4$ and $\ell = 2$ and let us consider two 4-multipartition $\vec{\lambda} = ((2,1),(0),(0),(1))$ and $\vec{\mu} = ((0),(2,1),(1),(0))$. We have

$$T_{\vec{\lambda}} = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\, ,\ \emptyset\ ,\ \emptyset\ ,\ \boxed{4} \right) \quad \text{and} \quad T_{\vec{\mu}} = \left(\ \emptyset\ ,\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\, ,\ \boxed{4}\ ,\ \emptyset\ \right)$$

We see that $r(T_{\vec{\lambda}}) = r(T_{\vec{\mu}}) = (2,3,1,2)$ and therefore $u = u_{\vec{\lambda}} = u_{\vec{\mu}}$ will be



Its associated matrix factorization is $\widehat{u} = \widehat{F}_{4,2,(3,4,0,1)}\widehat{F}_{3,1,(4,3,0,1)}\widehat{F}_{2,3,(4,3,1,0)}\widehat{F}_{1,2,(4,4,0,0)}$. The idempotent for both 4-multipartitions is therefore the identity homomorphism $\mathrm{id}\colon \widehat{u} \to \widehat{u}$. But the "dot placement" will be different, because $\vec{\lambda}$ has only three addable nodes for the first F_2, while $\vec{\mu}$ has two addable nodes for the first F_2 and one for the second. Thus, we have

$$e(\vec{\lambda})d(\vec{\lambda}) = \widehat{t}_1^{\,3}\colon \widehat{u} \to \widehat{u} \ \text{ and } \ e(\vec{\mu})d(\vec{\mu}) = \widehat{t}_4\widehat{t}_1^{\,2}\colon \widehat{u} \to \widehat{u}.$$

5.1.3. *The symmetric group and homomorphisms of matrix factorizations.* We are going to show now how to use the symmetric group to define homomorphisms between matrix factorizations.

Remark 5.5. Fix a n-multipartition $\vec{\lambda}$ with $c(\vec{k})$ nodes. Recall that the set $\mathrm{Std}(\vec{\lambda})$ denotes the set of *all* standard fillings of $\vec{\lambda}$. Now the symmetric group $S_{c(\vec{k})}$ makes its appearance because it acts on

the subset $\mathrm{Std}_1(\vec{\lambda})$ of all standard fillings where every entry appears *just once*. The action for the simple transpositions τ_k is defined to *exchange (if possible) k and $k+1$*.

Moreover, it should be noted that $S_{c(\vec{k})}$ acts on the set of strings of F's of length $c(\vec{k})$ with a *fixed number of occurrences of the F's* by defining the action of the k-th transposition τ_k by *exchanging the neighboring entries k and $k+1$* reading from right to left (as usual). In order to remember the residue as well, we denote a transposition τ_k that exchanges a F_i and a $F_{i'}$ by $\tau_k(i,i')$, that is

$$\tau_k(i,i')(F_{c(\vec{k})}\cdots \underbrace{F_{i'}F_i}_{\text{pos. } k}\cdots F_1) = F_{c(\vec{k})}\cdots \underbrace{F_iF_{i'}}_{\text{pos. } k}\cdots F_1.$$

Note that these two actions agree. To see this recall that, by our discussion in Section 4.1, an element of $\mathrm{Std}_1(\vec{\lambda})$ gives rise to a string of F_i's by reading the nodes ordered by their number and turn them into a string of F_i's by setting the i for the k-th (from right to left) F to be the residue of the node with label k.

We define $\tau_k(i,i')^* = \tau_k(i',i)$ and $\sigma^* = (\tau_{k_r}(i_r,i'_r)\cdots\tau_{k_1}(i_1,i'_1))^* = \tau_{k_1}(i'_1,i_1)\cdots\tau_{k_r}(i'_r,i_r)$.

Definition 5.6. (**Homomorphisms between matrix factorizations**) Given two strings of F's

$$\mathrm{qH}_1 = \prod_{k=1}^{c(\vec{k})} F_{i_k} = F_{i_{c(\vec{k})}}\cdots F_{i_1} \quad \text{and} \quad \mathrm{qH}_2 = \prod_{k=1}^{c(\vec{k})} F_{i'_k} = F_{i'_{c(\vec{k})}}\cdots F_{i'_1}.$$

Let $\widehat{u}_1$ and $\widehat{u}_2$ denote the two matrix factorizations that we associate to the corresponding $\mathfrak{sl}_n$-webs $u_1 = \mathrm{qH}_1 v_{(n^\ell)}$ and $u_2 = \mathrm{qH}_2 v_{(n^\ell)}$.

We assume that qH_1 and qH_2 differ only by a permutation $\sigma \in S_{c(\vec{k})}$ of their F's and that σ is already decomposed into a string of transpositions

$$\sigma = \tau_{k_l}\ldots\tau_{k_1},$$

such that $\sigma\cdot\mathrm{qH}_1 = \mathrm{qH}_2$. Then we associate to the triple qH_1, qH_2 and σ a *homomorphism of matrix factorizations*

$$\phi_\sigma(\mathrm{qH}_1,\mathrm{qH}_2)\colon \widehat{u}_1\to\widehat{u}_2,\quad \phi_\sigma(\mathrm{qH}_1,\mathrm{qH}_2) = \phi(\tau_{k_l}(i_l,i'_l))\circ\cdots\circ\phi(\tau_{k_1}(i_1,i'_1))\circ\mathrm{Id}_{\widehat{u}_1} \tag{5.1.1}$$

by composing the identity $\mathrm{Id}_{\widehat{u}_1}$ on $\widehat{u}_1$ from the left with the homomorphisms of matrix factorizations

$$\phi(\tau_{k_r}(i_r,i'_r)) \mapsto \begin{cases} \widehat{CR}_{k_r,i_ri_r\pm 1}, & \text{if } i'_r = i_r\pm 1,\\ \widehat{I}_{k_r,i_ri_r}\widehat{D}_{k_r,i_ri_r}, & \text{if } i_r = i'_r,\\ \widehat{s}_{k_r,i_ri'_r}, & \text{if } |i_r - i'_r| > 1,\end{cases}$$

where all the other parts should be the identity. Note that this procedure really gives a homomorphism of matrix factorizations from $\widehat{u}_1$ to $\widehat{u}_2$.

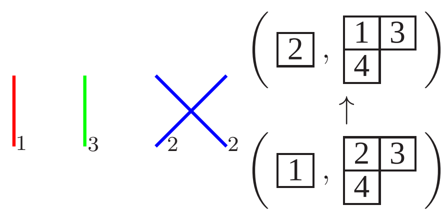
We note that this depends on the choice of the decomposition of σ into transpositions. We *choose* a certain decomposition in the following. We should point out that this choice only makes sense in a special case where the n-multitableaux $\vec{T}_1$ and $\vec{T}_2$ associated to u_1 and u_2 are of the *same shape* $\vec{T}_1,\vec{T}_2 \in \mathrm{Std}_1(\vec{\lambda})$ for some $\vec{\lambda}$. Moreover, since we always want to factor through an idempotent associated with something "canonical", we only fix such a decomposition for the special case where $\vec{T}_2 = T_{\vec{\lambda}}$ (recall that $T_{\vec{\lambda}}$ was defined in Definition 3.7).

That is, for a fixed n-multipartition $\vec{\lambda}$ and a corresponding n-multitableau $\vec{T} \in \mathrm{Std}_1(\vec{\lambda})$, we choose a fixed permutation $\sigma \in S_{c(\vec{k})}$ such that

$$\sigma \cdot \vec{T} = T_{\vec{\lambda}}$$

by searching for the lowest $k \in \{1, \ldots, c(\vec{k})\}$ such that the node N with entry k in $\vec{T}$ is not the same as the node N' with entry k in $T_{\vec{\lambda}}$. Apply a minimal sequence of transpositions until they match and repeat the process until $\sigma \cdot \vec{T} = T_{\vec{\lambda}}$. By construction, the permutation $\sigma \in S_{c(\vec{k})}$ will be of minimal length with respect to the property $\sigma \cdot \vec{T} = T_{\vec{\lambda}}$. We denote the homomorphism of matrix factorizations associated to this permutation σ by ϕ_σ.

We point out that this combinatorial construction of the homomorphisms (which can be thought of a sophisticated version of Specht theory) can not be read of from a (cyclotomic) KL-R diagram directly as the following example illustrates.



In the example above the two $\mathfrak{sl}_2$-webs u_1, u_2 are the same $u_1 = u_2 = F_1F_3F_2F_2v_{(2^1)}$ and there is a non-trivial diagram that we can not see by just looking at the boundary. But one can associate different 2-multitableaux to them, as illustrated above. So we need the language of n-multitableaux.

It is worth noting that this procedure is well-defined, i.e. one does not run into ambiguities and the resulting homomorphism is between $\widehat{u}_1$ and $\widehat{u}_2$, by Remark 5.5.

Furthermore, it is easy to see that

$$\sigma \cdot \vec{T}_1 = \vec{T}_2 \Leftrightarrow \sigma^* \cdot \vec{T}_2 = \vec{T}_1 \text{ for all } \vec{T}_1, \vec{T}_2 \in \mathrm{Std}_1(\vec{\lambda}),\ \sigma \in S_{c(\vec{k})}.$$

Lemma 5.7. *Given the set-up from Definition 5.6. Then there exists an element in R_Λ, also denoted by $\phi_\sigma(\mathrm{qH}_1, \mathrm{qH}_2)$, such that*

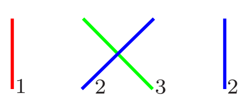
$$\tilde{\Gamma}\colon \phi_\sigma(\mathrm{qH}_1, \mathrm{qH}_2) \in R_\Lambda \mapsto \phi_\sigma(\mathrm{qH}_1, \mathrm{qH}_2).$$

Proof. The proof works essentially as the proof of Lemma 5.3, i.e. we show that there exists an element of the KL-R part of $\mathcal{U}(\mathfrak{sl}_m)$, that we denote again by the same expression, such that

$$\Gamma_{m,n\ell,n}\colon \phi_\sigma(\mathrm{qH}_1, \mathrm{qH}_2) \in \mathcal{U}(\mathfrak{sl}_m) \mapsto \phi_\sigma(\mathrm{qH}_1, \mathrm{qH}_2).$$

Comparing again Rouquier's definition before Lemma 5.4 in [65] to our convention, we see that this proves the lemma.

The element of the KL-R part of $\mathcal{U}(\mathfrak{sl}_m)$ is obtained by putting the string of F's for qH_1 at the bottom and the one for qH_2 at the top and then draw a diagram consisting of crossings given by the procedure from 5.6 in between. For example



$$\rightsquigarrow \tau_2(3,2)\colon F_1F_2F_3F_2 \to F_1F_3F_2F_2.$$

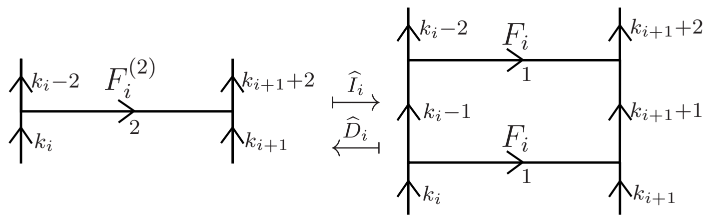
This show the existence of the $\phi_\sigma(\mathrm{qH}_1, \mathrm{qH}_2) \in R_\Lambda$ we need. $\square$

5.1.4. *The categorified growth algorithm.* We are now able to give the definition of the categorification of our extended growth algorithm.

To define the basis for the $\mathfrak{sl}_n$-web algebra ${}_vH_n(\vec{k})_u$ for any $\mathfrak{sl}_n$-webs u and v we need to use certain isomorphisms of matrix factorizations between the left and right side of the square removal 3.2.6. That is, we have to go to the thick cyclotomic KL-R $\check{R}_\Lambda$ from Definition 3.32 and have to associate something to the split and merge from Subsection 3.3.1.

Thus, we need to substitute all divided powers $F_i^{(j)}$ in the sequence associated to u to j-times F_i. This means in pictures that we replace (here $j = 2$)

(5.1.2)



The morphisms $\widehat{I}_i$ and $\widehat{D}_i$ are *not* isomorphisms and of q-degree -1 - as the split and merge. Thus, we have to choose: Starting with a flow on the right picture in 5.1.2, we choose *one* flow for the left. Our choice will ensure that the whole process preserves the q-degree because the chosen flow will be of weight one lower than the starting flow. The precise definition of $\widehat{I}_i$ and $\widehat{D}_i$ are not important for us (and long) and can be found for example in [55] Definition 8.11 and 8.12.

For $j > 2$ we do literally the same, but use the image under $\check{\Gamma}_{m,n\ell,n}$ (see Theorem 3.36) of the splitters $\mathcal{F}_i^{(j')} \to \mathcal{F}_i^{(j'-1)}\mathcal{F}_i$ repeatedly. We denote them by $\widehat{I}_i^{j'}$. These are of degree $-j'+1$. Thus, the full splitter is of degree $-(j + j - 1 + j - 2 + \cdots + 1)$. Our choice in this case will ensure that the whole process preserves the q-degree because the chosen flow will be of weight $j + j - 1 + j - 2 + \cdots + 1$ lower than the starting flow.

Definition 5.8. (**Homomorphism of matrix factorizations for $\mathfrak{sl}_n$-webs u_f with a flow**) Given a $\mathfrak{sl}_n$-web with a flow u_f, we associate to it a homomorphism of matrix factorizations

$$\phi_{u_f} \colon \widehat{u}_f \to \widehat{u}_{\vec{\lambda}},$$

where $\vec{\lambda}$ is the boundary datum/n-multipartition and $\widehat{u}_{\vec{\lambda}}$ is as in 5.1, in the following way.

Change the n-multitableau $\iota(u_f)$ by replacing the lowest multiple entry k of multiplicity j_k of $\iota(u_f)$ *increasing* from left to right with consecutive numbers $k, \ldots, k+j_k$ and shift all other entries by j_k. Repeat until no multiple entries occur and obtain $\iota(u_f)'$ (see also Example 3.9). Set

$$\phi_{u_f} = \phi_\sigma(\iota(u_f)', T_{\vec{\lambda}}) \circ \phi_{\mathrm{R}} \colon \widehat{u}_f \to \widehat{u}_{\vec{\lambda}},$$

with $\phi_\sigma(\iota(u_f)', T_{\vec{\lambda}})$ for the strings of F_i's $\mathrm{qH}_{1,2}$ corresponding to $\iota(u_f)'$ and $T_{\vec{\lambda}}$ respectively.

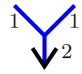
The homomorphism ϕ_{R} is given by composing an appropriate number of the $\widehat{I}$'s from below. That is, the difference between the two corresponding $\mathfrak{sl}_n$-webs is $\ldots F_i^{(j)} \ldots$ for $\widehat{u}_f$ and $\ldots F_i \cdots F_i \ldots$ for $\phi_\sigma(\iota(u_f)', T_{\vec{\lambda}})$ which are replaced inductively by $\widehat{I}_i^{j'}$'s: The order does *not matter* by the associativity of splitters (Proposition 2.2.4 in [42]) combined with Theorem 3.36.

Lemma 5.9. *There is a diagram in* $\check{\mathcal{U}}(\mathfrak{sl}_m)$*, denoted by the same symbol, such that*

$$\check{\Gamma}_{m,n\ell,n} \colon \phi_{u_f} \mapsto \phi_{u_f},$$

where $\check{\Gamma}_{m,n\ell,n}$ *is the extended functor from Theorem 3.36.*

Proof. It follows from our construction that (the color should be i)

$$\check{\Gamma}_{m,n\ell,n}\colon$$  $$\mapsto \widehat{I}_i\colon F_i^{(2)} \to F_i^1 F_i^1.$$

This induces maps for all the thick splitters and shows that ϕ_R comes from a diagram in $\check{\mathcal{U}}(\mathfrak{sl}_m)$. We can use Lemma 5.7 to see that $\phi_\sigma(\iota(u_f)', T_{\vec{\lambda}})$ comes from a diagram in $\mathcal{U}(\mathfrak{sl}_m)$. Combining both we obtain the statement. □

We are now able go state a *growth algorithm for homomorphism of matrix factorizations* which gives rise to a graded cellular basis.

Definition 5.10. (**Growth algorithm for homomorphisms of matrix factorizations**) Let us denote by $B(W_n(\vec{k}))$ any monomial basis of the $\mathfrak{sl}_n$-web space $W_n(\vec{k})$. We denote by $\tilde{B}(W_n(\vec{k}))$ the set of all basis elements together with a choice of a flow line. We note that, because we have chosen a basis, none of the $\mathfrak{sl}_n$-webs in $\tilde{B}(W_n(\vec{k}))$ will be isotopic.

Given a state string $\vec{S}$, the corresponding n-multipartition $\vec{\lambda}$ and $u_f, v_{f'} \in \tilde{B}(W_n(\vec{k}))$. We define a homomorphism following Definition 5.8 by

$$\mathcal{F}^{\vec{\lambda}}_{\iota(v_{f'}),\iota(u_f)}\colon \widehat{u} \to \widehat{v}, \qquad \mathcal{F}^{\vec{\lambda}}_{\iota(v_{f'}),\iota(u_f)} = \phi^*_{v_{f'}} e(\vec{\lambda}) d(\vec{\lambda}) \phi_{u_f},$$

where the * for $\phi_{v_{f'}}$ is defined as

$$\phi^*_{v_{f'}} = (\phi_{v_{f'}})^* = (\phi_\sigma(\iota(v_{f'})', T_{\vec{\lambda}}) \circ \phi_{\mathrm{R}})^* = \phi^*_{\mathrm{R}} \circ \phi_{\sigma^*}(T_{\vec{\lambda}}, \iota(v_{f'})').$$

Here the ϕ^*_{R} consists of $\widehat{D}_i^{j'}$'s going in the other direction than the corresponding $\widehat{I}_i^{j'}$'s, see 5.1.2.

Lemma 5.11. *There is a diagram in* $\check{\mathcal{U}}(\mathfrak{sl}_m)$*, denoted by the same symbol, such that*

$$\check{\Gamma}_{m,n\ell,n}\colon \mathcal{F}^{\vec{\lambda}}_{\iota(v_{f'}),\iota(u_f)} \mapsto \mathcal{F}^{\vec{\lambda}}_{\iota(v_{f'}),\iota(u_f)}.$$

Moreover, if $\iota(u_f) = \iota(u_f)'$ *and* $\iota(v_{f'}) = \iota(v_{f'})'$*, then there is an element of the HM-basis of* R_Λ*, denoted by the same symbol, such that*

$$\tilde{\Gamma}\colon \mathcal{F}^{\vec{\lambda}}_{\iota(v_{f'}),\iota(u_f)} \mapsto \mathcal{F}^{\vec{\lambda}}_{\iota(v_{f'}),\iota(u_f)}.$$

This element is completely determined by $u_f, v_{f'}$ *in the sense that changing either the* $\mathfrak{sl}_n$*-webs or the flows will give another element of the HM-basis.*

Proof. The first and second statement are just combinations of Lemmata 5.3, 5.7 and 5.9. The third statement follows from our translation in Section 4.1, i.e. the HM-basis element $\psi^{\vec{\lambda}}_{\vec{T}',\vec{T}}$ (see Definition 3.3.3 or, with a slightly different notation, Definition 5.1 in [30]) with the datum

$$(\vec{\lambda}, \vec{T} = \iota(u_f)' \in \mathrm{Std}(\vec{\lambda}), \vec{T}' = \iota(v_{f'})' \in \mathrm{Std}(\vec{\lambda}))$$

will be the one for $\mathcal{F}^{\vec{\lambda}}_{\iota(v_{f'}),\iota(u_f)}$. □

Remark 5.12. One can show analogously as the author has done in Lemma 4.15 of [75] that

$$\deg_q(\mathcal{F}^{\vec{\lambda}}_{\iota(v_{f'}),\iota(u_f)}) = \deg_{\mathrm{wt}}(u_f) + \deg_{\mathrm{wt}}(v_{f'}) = \deg_{\mathrm{BKW}}(\iota(u_f)) + \deg_{\mathrm{BKW}}(\iota(v_{f'})).$$

The main ingredient is of course the translation from Proposition 4.12. The reader should be careful, because the homomorphisms ϕ_R are not of degree zero. But our convention to obtain $\iota(u_f)'$ from $\iota(u_f)$ ensures that the shift of degree is exactly the difference of the degrees of $\iota(u_f)'$ and $\iota(u_f)$.

Example 5.13. As a small example consider the $\mathfrak{sl}_4$-web from Example 5.4. We use the growth algorithm to generate elements of ${}_uH_n(\vec{k})_u$. Note that we have to choose a flow in order to give an example and, in this very special case, the flow only depends on its boundary datum. Thus, everything will be symmetric and the flows can be read of from the cut-line.

The two flows that belong to $T_{\vec{\lambda}}$ and $T_{\vec{\mu}}$ are given by $S_{\vec{\lambda}} = (\{3,2,1\},\{4,3,2\},\{1\},\{4\})$ and $S_{\vec{\mu}} = (\{4,2,1\},\{4,3,1\},\{2\},\{3\})$ respectively. In these two cases the corresponding elements are just given by the dotted idempotents from Example 5.4, since we do not have to let the symmetric group S_4 act on the 4-multitableaux.

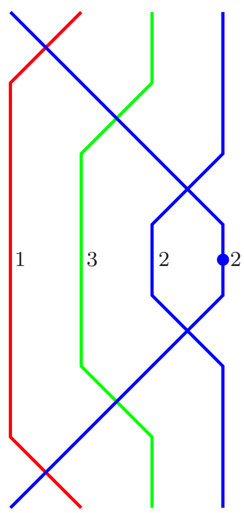
The flow f, given by $S = (\{4,3,2\},\{4,3,1\},\{2\},\{1\})$, on the other hand gives rise to

$$\iota(u_f) = \left(\emptyset\,,\ \emptyset\,,\boxed{4}\,,\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}\right) \quad \text{and} \quad T_{\vec{\lambda}_f} = \left(\emptyset\,,\ \emptyset\,,\boxed{1}\,,\ \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & \\ \cline{1-1}\end{array}\right).$$

Thus, the permutation $\tau_1(2,2)\tau_2(3,2)\tau_3(1,2)$ gives $\tau_1(2,2)\tau_2(3,2)\tau_3(1,2)\cdot\iota(u_f) = T_{\vec{\lambda}_f}$. In this case we see, since $\phi_R = \mathrm{id}$ and $e(\vec{\lambda}_f)d(\vec{\lambda}_f) = \widehat{t}_1$, that

$$\mathcal{F}^{\vec{\lambda}_f}_{\iota(u_f),\iota(u_f)} = \widehat{CR}_{3,21}\circ\widehat{CR}_{2,23}\circ\widehat{I}_{2,22}\widehat{D}_{2,22}\circ\widehat{t}_1\circ\widehat{I}_{2,22}\widehat{D}_{2,22}\circ\widehat{CR}_{3,32}\circ\widehat{CR}_{3,12}\colon \widehat{u}\to\widehat{u}$$

where the degree is $\deg(\mathcal{F}^{\vec{\lambda}_f}_{\iota(u_f),\iota(u_f)}) = 2 = \deg_{\mathrm{BKW}}(\iota(u_f)) + \deg_{\mathrm{BKW}}(\iota(u_f))$. Moreover, we invite the reader to verify that the corresponding element in the (thick) cyclotomic KL-R is



$$\begin{aligned}
&\longleftrightarrow \tau_3(2,1)\\
&\longleftrightarrow \tau_2(2,3)\\
&\longleftrightarrow \tau_1(2,2)\\
&\longleftrightarrow e(\vec{\lambda}_f)d(\vec{\lambda}_f)\\
&\longleftrightarrow \tau_1(2,2)\\
&\longleftrightarrow \tau_2(3,2)\\
&\longleftrightarrow \tau_3(1,2)
\end{aligned}$$

It is worth noting that one has a "dot-bursting" because $\widehat{D}_{2,22}\circ\widehat{t}_1\circ\widehat{I}_{2,22} = \pm\mathrm{id}$ is a KL-R relation mentioned below in Remark 5.17. This is a phenomena that happens rather often. In fact, the "dot placement" is the part that comes purely from the cyclotomic Hecke algebra and is somehow mysterious in the $\mathfrak{sl}_n$-web framework.

5.1.5. *It is a basis!* We are now able to proof that the growth algorithm given in Definition 5.10 gives a basis of the $\mathfrak{sl}_n$-web algebra ${}_vH_n(\vec{k})_u$. The main ingredients are the results from Section 4.1.

Theorem 5.14. *The growth algorithm from Definition 5.10 gives a homogeneous basis of* ${}_vH_n(\vec{k})_u$.

Proof. We will show that the growth algorithm gives a linear independent set denoted by

$$\mathfrak{F} = \{\mathcal{F}^{\vec{\lambda}}_{\iota(v_{f'}),\iota(u_f)} \in {}_vH_n(\vec{k})_u \mid (\vec{S}, u_f, v_{f'}),\ \vec{S}\text{ is a state string},\ u_f, v_{f'} \in \tilde{B}(W_n(\vec{k}))\}.$$

By a counting argument, which heavily relies on the translation from Section 4.1, we see that this set forms a basis, since we know that the set of all triples

$$(\vec{\lambda}, \iota(u_f) \in \mathrm{Std}(\vec{\lambda}), \iota(v_{f'}) \in \mathrm{Std}(\vec{\lambda}))$$

has the same size as a possible basis of ${}_vH_n(\vec{k})_u$: The set of all possible flows on v^*u has the same size as $\dim(\mathrm{EXT}(\widehat{u}, \widehat{v}))$ since the Euler form $\dim(\mathrm{EXT}(\cdot,\cdot))$ categorifies the Kuperberg bracket (which can be deduced from results in Sections 6 to 11 in [79] or Section 3 in [80], i.e. that matrix factorizations satisfy the $\mathfrak{sl}_n$-web relations). Hence, we conclude that the linear independence of $\mathfrak{F}$ suffices to show that the set $\mathfrak{F}$ forms a basis.

We want to consider the additive equivalence of 2-categories $\tilde{\Gamma}$ from Theorem 3.35. The argument goes as follows. The linear independence of the set

$$\mathfrak{F}' = \{\mathcal{F}^{\vec{\lambda}}_{\iota(v_{f'})', \iota(u_f)'} \in {}_vH_n(\vec{k})_u \mid (\vec{S}, u_f, v_{f'}),\ \vec{S} \text{ is a state string},\ u_f, v_{f'} \in \tilde{B}(W_n(\vec{k}))\},$$

that is, without the removals ϕ_R, suffices to show that $\mathfrak{F}$ is also linear independent. To see this note that the homomorphisms from Equation 5.1.2 give rise to an isomorphism between the left side and a q-shifted sum of the right side (they correspond to the splitters and merges and the isomorphism can be verified as in Theorem 5.1.1 of [42]). Our choice of ϕ_R is a restriction of this isomorphism to a certain summand (and forget the q-degree shift).

But the set $\mathfrak{F}'$ comes, by our translation from Section 4.1 and Lemma 5.11, directly from a (usually strict!) subset $\mathfrak{F}'_{\mathrm{HM}}$ of the HM-basis in some cyclotomic KL-R algebra, i.e. we have

$$\tilde{\Gamma}(\mathfrak{F}'_{\mathrm{HM}}) = \mathfrak{F}' \quad \text{and} \quad |\mathfrak{F}'_{\mathrm{HM}}| = |\mathfrak{F}'|.$$

Since $\tilde{\Gamma}$ is an additive equivalence of 2-categories and all subsets of the HM-basis are linear independent, we see that $\mathfrak{F}'$ has to be linear independent, too.

Hence, the set $\mathfrak{F}$ is linear independent and therefore, by the counting argument mentioned above, also spanning, i.e. it is a basis. This basis is clearly homogeneous by our construction as a composition of some generators of a certain degree. □

We immediately obtain the following corollary, since

$$H_n(\vec{k}) = \bigoplus_{u,v \in B(W_n(\vec{k}))} {}_vH_n(\vec{k})_u \quad \text{and} \quad H_n(\Lambda) = \bigoplus_{\vec{k} \in \Lambda(m, n\ell)_n} H_n(\vec{k}).$$

Corollary 5.15. *The growth algorithm from Definition 5.10 gives a homogeneous basis of $H_n(\vec{k})$ and of $H_n(\Lambda)$ respectively.* □

In order to connect the $\mathfrak{sl}_n$-web algebras to the thick cyclotomic KL-R $\check{R}_\Lambda$, we define

$$\check{R}(\vec{k}) = \bigoplus_{u,v \in B(W_n(\vec{k}))} e(\vec{\lambda}^v_c)\check{R}_\Lambda e(\vec{\lambda}^u_c) \quad \text{and} \quad \check{R}(\Lambda) = \bigoplus_{\vec{k} \in \Lambda(m, n\ell)_n} \check{R}(\vec{k}),$$

where $\vec{\lambda}^u_c$ denotes the canonical n-multipartition (see Definition 4.16) associated to u and $e(\vec{\lambda}^u_c)$ is the associated idempotent from Lemma 5.3.

Theorem 5.16. *Let $u, v \in W_n(\vec{k})$ be two $\mathfrak{sl}_n$-webs. Then*

$$e(\vec{\lambda}^v_c)\check{R}_\Lambda e(\vec{\lambda}^u_c) \cong {}_vH_n(\vec{k})_u \textit{ (graded).}$$

This gives rise to isomorphisms of graded algebras

$$\check{R}(\vec{k}) \cong H_n(\vec{k}) \quad \textit{and} \quad \check{R}(\Lambda) \cong H_n(\Lambda)$$

which extends 3.3.5 to an additive equivalence of 2*-categories*

$$\check{\Gamma}\colon\ \check{\mathrm{R}}_\Lambda\text{-p}\boldsymbol{Mod}_{\mathrm{gr}} \to \mathcal{W}^p_\Lambda,$$

i.e. from the category of finite dimensional, $\mathbb{Z}$*-graded, projective* $\check{R}_\Lambda$*-modules to* $\mathcal{W}^p_\Lambda$.

Proof. This is just an assembling of pieces now. By Lemma 5.11 the basis of ${}_vH_n(\vec{k})_u$ that we have obtained in Theorem 5.14 comes from a set of the same size in $\check{\mathcal{U}}(\mathfrak{sl}_m)$ via our extension of the categorified q-skew Howe duality from Theorem 3.36. By the faithfulness of $\tilde{\Gamma}$ from Theorem 3.35 and the fact that the ϕ_R's come from certain compositions of splitters and merges, we get an inclusion of graded $\bar{\mathbb{Q}}$-vector spaces

$$e(\vec{\lambda}^v_c)\check{R}_\Lambda e(\vec{\lambda}^u_c) \hookrightarrow {}_vH_n(\vec{k})_u.$$

Thus, a counting argument can ensure again that they are isomorphic. The graded dimension of the left side is known by Theorem 4.10 in [7]. Using our results from Proposition 4.12, we see that the graded dimensions are the same, since the right side's graded dimension (up to a shift) can be obtained by counting all weights of flows on v^*u (as already explained in the proof of Theorem 5.14). Thus, we get an isomorphism.

The other statements are now just direct consequences of the first isomorphism. □

Remark 5.17. We should note here (already with the computation method from Section 5.2 in mind) that it follows from Theorem 5.16 that the homomorphisms $\mathcal{F}^{\vec{\lambda}}_{\iota(v_{f'}),\iota(u_f)}$ are *local* in the sense that all their factors satisfy the thick cyclotomic KL-R relations. One can use these local relations to re-write the homomorphisms in a (at least for a machine) not to complicated way. A list of these relations can be found in different places, e.g. either using diagrams in [38], [39] or as an algebraic list in [30]. Moreover, a list of local rules for the thick cyclotomic KL-R can be deduced from the ones for splitters and merges given in Section 2 of [42].

Remark 5.18. The definition of the * gives rise to an involution on the $\mathfrak{sl}_n$-web algebra $H_n(\vec{k})$ by Theorem 5.14 and a small calculation that

$$(\mathcal{F}^{\vec{\lambda}}_{\iota(v_{f'}),\iota(u_f)})^* = \mathcal{F}^{\vec{\lambda}}_{\iota(u_f),\iota(v_{f'})}.$$

It is worth noting that this is exactly the involution Mackaay defines before Remark 7.8 in [50] using Brundan and Kleshchev's duality on the category of finite dimensional, projective modules of the cyclotomic KL-R algebra. His definition is not explicit as Mackaay points out himself. Our definition can, on the other hand, be computed explicitly.

5.1.6. *Cellularity.* The basis $\mathfrak{F}$ is a graded cellular basis of $H_n(\vec{k})$. Let us shortly recall the definition which is in the ungraded setting due to Graham and Lehrer [29] and in the graded setting to Hu and Mathas [30].

Definition 5.19. (**Graham-Lehrer, Hu-Mathas**) Suppose A is a $\mathbb{Z}$-graded free algebra over R of finite rank. A $\mathbb{Z}$*-graded cell datum* is an ordered quintuple $(\mathfrak{P}, \mathcal{T}, C, \mathrm{i}, \deg)$, where $(\mathfrak{P}, \rhd)$ is the *weight poset*, $\mathcal{T}(\lambda)$ is a finite set for all $\lambda \in \mathfrak{P}$, i is an *involution* of A and C is an injection

$$C\colon \coprod_{\lambda\in\mathfrak{P}} \mathcal{T}(\lambda) \times \mathcal{T}(\lambda) \to A,\ (s,t) \mapsto c^\lambda_{st}.$$

Moreover, the *degree function* deg is given by

$$\deg\colon \coprod_{\lambda\in\mathfrak{P}} \mathcal{T}(\lambda) \to \mathbb{Z}.$$

The whole data should be such that the c^{λ}_{st} form a homogeneous R-basis of A with $\mathrm{i}(c^{\lambda}_{st}) = c^{\lambda}_{ts}$ and $\deg(c^{\lambda}_{st}) = \deg(s) + \deg(t)$ for all $\lambda \in \mathfrak{P}$ and $s,t \in \mathcal{T}(\lambda)$. Moreover, for all $a \in A$

$$(5.1.3)\qquad ac^{\lambda}_{st} = \sum_{u\in\mathcal{T}(\lambda)} r_a(s,u)c^{\lambda}_{ut} \ (\mathrm{mod}\ A^{\rhd\lambda}).$$

Here $A^{\rhd\lambda}$ is the R-submodule of A spanned by the set $\{c^{\mu}_{st} \mid \mu \rhd \lambda \text{ and } s,t \in \mathcal{T}(\mu)\}$.

An algebra A with such a quintuple is called a *graded cellular algebra* and the c^{λ}_{st} are called a *graded cellular basis* of A (with respect to the involution i).

Theorem 5.20. ***(Graded cellular basis)*** *The algebra $H_n(\vec{k})$ is a graded cellular algebra in the sense of Definition 5.19 with the cell datum*

$$(5.1.4)\qquad (\mathfrak{P}^n_{c(\vec{k})}, \iota(\tilde{B}(W_n(\vec{k}))), \mathfrak{F}, {}^*, \deg_{\mathrm{BKW}}),$$

where $\mathfrak{P}^n_{c(\vec{k})}$ is the set of all n-multipartitions of $c(\vec{k})$ ordered by the dominance order $\rhd$ from Definition 3.7, $\iota(\tilde{B}(W_n(\vec{k})))$ is the image under our translation from Definition 4.2, the involution * is as above in Remark 5.18 and the degree $\deg_{\mathrm{BKW}}$ on the n-multitableaux in $\iota(\tilde{B}(W_n(\vec{k})))$. These cell data (one for each $\vec{k} \in \Lambda(m, n\ell)_n$) can be extended to $H_n(\Lambda)$.

We think it is worthwhile to note here that our proof of the cellularity below uses the HM-basis and therefore in the end the combinatorics of the cyclotomic Hecke algebra: Hu and Mathas's result relies on the Dipper, James and Mathas standard basis (see [24]) and thus, on n-multitableaux combinatorics. The extra part we need to verify comes from thick calculus and can be easier seen in the diagrammatic language of the thick cyclotomic KL-R (which we use below). The $\mathfrak{sl}_n$-web algebra framework is on the other hand more useful to see connections with topology, e.g. the $\mathfrak{sl}_n$-link homologies. Thus, we think all three perspectives are useful.

Proof. To shorten our notation we skip the $\iota(\cdot)$ in the following. Moreover, the scalars below should all depend on the left side of the multiplication.

We have to prove four statements to show that 5.1.4 is a graded cell datum for $H_n(\vec{k})$. The four statements are that $\mathfrak{F}$ is a basis of the graded algebra $H_n(\vec{k})$, the elements $\mathcal{F}^{\vec{\lambda}}_{v_{f'},u_f} \in \mathfrak{F}$ are homogeneous of degree

$$\deg_q(\mathcal{F}^{\vec{\lambda}}_{v_{f'},u_f}) = \deg_{\mathrm{BKW}}(u_f) + \deg_{\mathrm{BKW}}(v_{f'}),$$

the involution * satisfies

$$(\mathcal{F}^{\vec{\lambda}}_{v_{f'},u_f})^* = \mathcal{F}^{\vec{\lambda}}_{u_f,v_{f'}}$$

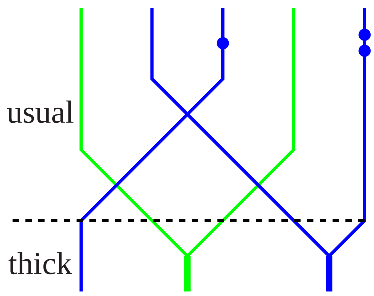
and the crucial (which suffices to verify Equation 5.1.3 by linearity)

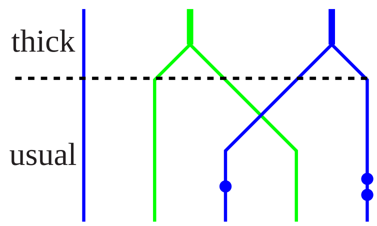
$$(5.1.5)\qquad \mathcal{F}^{\vec{\mu}}_{\tilde{v}_{\tilde{f}'},\tilde{u}_{\tilde{f}}}\mathcal{F}^{\vec{\lambda}}_{v_{f'},u_f} = \sum_{w_{f''}\in\tilde{B}(W_n(\vec{k}))} r_{v_{f'},w_{f''}}\mathcal{F}^{\vec{\lambda}}_{w_{f''},u_f} \ (\mathrm{mod}\ H_n(\vec{k})^{\rhd\lambda}).$$

The first statement is just Corollary 5.15, the second follows from Remark 5.12 (which is based on Proposition 4.12) and the third follows almost directly from the definition of *, see Remark 5.18.

To verify Equation 5.1.5 we note that the product is zero if the two $\mathfrak{sl}_n$-webs $\tilde{u}$ and v are not the same. Thus, we can focus on the case $\tilde{u} = v$.

Since the "thick cellularity" can be easier seen in the thick cyclotomic KL-R set-up (to which we can freely switch by Theorem 5.16) let us illustrate with thick cyclotomic KL-R diagrams how we can prove Equation 5.1.5. Moreover, to prove Equation 5.1.5 it is enough to consider only the "middle" part (after the "dotted" idempotent $e(\vec{\lambda})d(\vec{\lambda})$ and before the "dotted" idempotent $e(\vec{\mu})d(\vec{\mu})$). Thus, this is the only part we illustrate below (the right diagram is the top of $e(\vec{\lambda})d(\vec{\lambda})$).



and


We have illustrated two typical examples above. Everything splits into a "usual" and a "thick" part.

The main point is that, by our construction from Definition 5.8, the assumption $\tilde{u} = v$ implies that the "thick" part of both are mirrors of each other. Thus, composing the two pictures will always create a composition of the split$\circ$merge as in Equation 3.3.1. This will always create extra crossings which are part of the "usual" story. Thus, it suffices to verify Equation 5.1.5 in the case of the cyclotomic KL-R algebra where we do not have any splitters or merges at all.

We can now use Lemma 5.11 and the proof of cellularity by Hu and Mathas, see Theorem 5.8 in [30], to see that Equation 5.1.5 holds in the "usual" cyclotomic KL-R set-up. We note that there are some technical points what kind of n-multitableaux can appear for a fixed basis. But it turns out that they do not matter. The proof of this is essentially the same as in the $\mathfrak{sl}_3$ case and can be directly adopted from the corresponding proof there (that is, the part after Equation 4.6 in the proof of Theorem 4.22 in [75]). Thus, using their result and the isomorphism (which preserves by Lemma 5.11 the dominance order $\rhd$) from Theorem 5.16, we see that Equation 5.1.5 is satisfied which finishes the proof. $\square$

Remark 5.21. We note that there is another convention to obtain "a HM-basis". That is, one could also use the "dual" n-multitableau $T^*_{\vec{k}}$ of $T_{\vec{k}}$ from Definition 3.7. Everything is this section works in the same vein as above. The difference is that the strings ϕ_σ of Definition 5.6 tend to be shorter for elements of low order but longer for elements of high order. We just have chosen to take the $T_{\vec{k}}$ to stay closer to Hu and Mathas formulation. This basis already appears in the non-thick form in Section 6 of [30] and they show in Theorem 6.11 that the dual basis is also cellular.

With respect to the $\mathfrak{sl}_n$-link homologies this dual basis has some advantages: It can be used to evaluate closed "$\mathfrak{sl}_n$-foams" as we show in Subsection 5.2.4. So let us shortly recall what the main differences in our set-up compared to Definition 5.10 are. There are only two, namely the following.

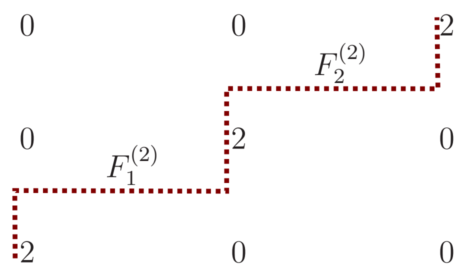
(1) The dotted idempotent $e(\vec{\lambda})d(\vec{\lambda})$ is obtained from "dual" n-multitableau $T^*_{\vec{k}}$ by counting addable boxes to the right. Same for the degree: Count addable and removable nodes *before* (to the *left*), see also Definition 3.1.
(2) We have to rearrange our conversion from Definition 5.8 for $\iota(u_f) \to \iota(u_f)'$ (recall that we needed this for the "thick" version) to $\iota(u_f) \to \tilde{\iota}(u_f)'$, where latter is obtained by replacing numbers *decreasing* from left to right instead of increasing from left to right.

A small example for (2) is the following.

$$\text{(usual)}\ (\boxed{1\,|\,3}, \boxed{2\,|\,4}) \leftarrow \vec{T} = (\boxed{1\,|\,2}, \boxed{1\,|\,2}) \to (\boxed{2\,|\,4}, \boxed{1\,|\,3})\ \text{(dual)}$$

The reason for this is just that our choice has to be different for the dual since the dual turns degrees and order around. Note that $\deg_{\mathrm{BKW}}(\vec{T}) = 0$ for both conventions due to our shift.

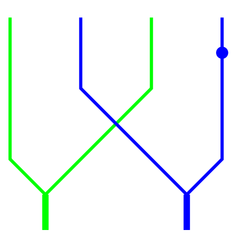
Example 5.22. Let us consider the following example. Compare also to Example 4.32. We want to illustrate the HM-basis for $\mathrm{EXT}(\widehat{u}, \widehat{v})$ for $n = 2$. The $\mathfrak{sl}_2$-web v should be the last one from Example 4.14 which is given by $F_2F_1F_2F_1v_{(2^1)}$. The other should be



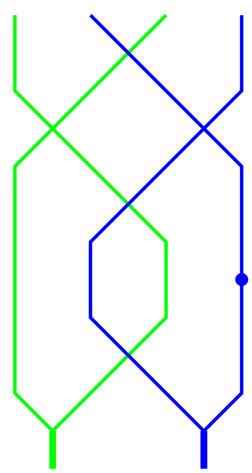
That is, $u = F_2^{(2)}F_1^{(2)}v_{(2^1)}$. The reader might think of elements of $\mathrm{EXT}(\widehat{u}, \widehat{v})$ as dotted cups and of $\mathrm{EXT}(\widehat{v}, \widehat{u})$ as dotted caps (in terms of Bar-Natan's cobordisms). As usual there is a duality: The dual of the un-dotted cup is the dotted cap. The same happens for the HM-basis and its dual.

We have one 2-multitableaux for u, namely $\vec{T}$ from Remark 5.21, and two for v, namely $\vec{T}_1$ and $\vec{T}_2$ from Example 4.32. The HM-basis for $\mathrm{EXT}(\widehat{u}, \widehat{v})$ is (using our isomorphism from Theorem 5.16) given by the two diagrams (of degree $\deg_{\mathrm{BKW}}(\vec{T}_1) = +1$ and $\deg_{\mathrm{BKW}}(\vec{T}_2) = -1$)

$(\boxed{1\,|\,2}, \boxed{3\,|\,4})$
$\tau_2(2,1)\uparrow$
$(\boxed{1\,|\,3}, \boxed{2\,|\,4})$
"un-thick"$\uparrow$
$(\boxed{1\,|\,2}, \boxed{1\,|\,2})$



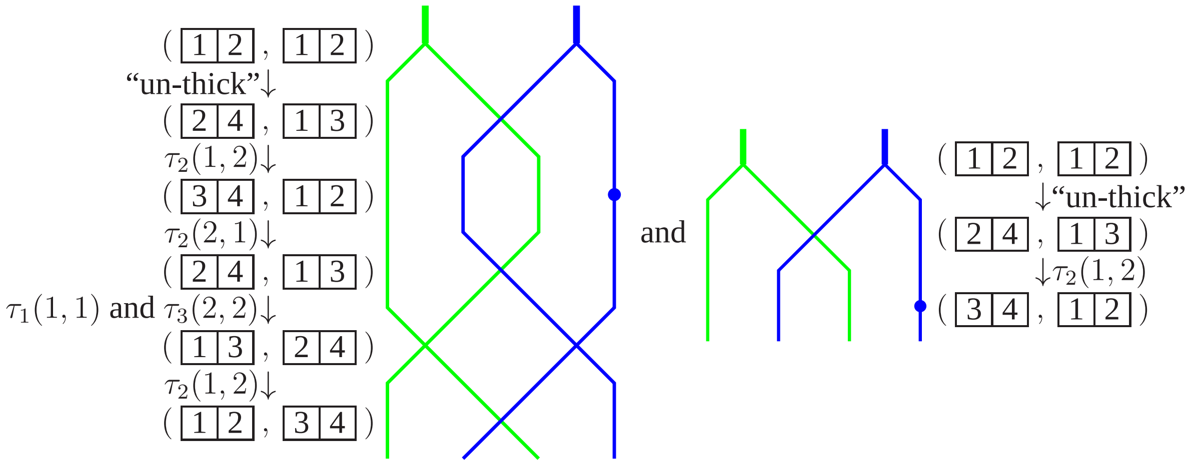
and



$(\boxed{3\,|\,4}, \boxed{1\,|\,2})$
$\uparrow\tau_2(1,2)$
$(\boxed{2\,|\,4}, \boxed{1\,|\,3})$
$\uparrow\tau_1(1,1)$ and $\tau_3(2,2)$
$(\boxed{1\,|\,3}, \boxed{2\,|\,4})$
$\uparrow\tau_2(1,2)$
$(\boxed{1\,|\,2}, \boxed{3\,|\,4})$
$\uparrow\tau_2(2,1)$
$(\boxed{1\,|\,3}, \boxed{2\,|\,4})$
$\uparrow$"un-thick"
$(\boxed{1\,|\,2}, \boxed{1\,|\,2})$

as the reader is invited to check. The left corresponds to the datum $(\vec{T}, \vec{T}_1)$ and the right one $(\vec{T}, \vec{T}_2)$. In the $\mathfrak{sl}_2$-cobordism language these are (up to a sign) just a dotted cup (left) and a cup

(right). The duals for $\mathrm{EXT}(\widehat{v}, \widehat{u})$ on the other hand are given by (of dual-degree $\deg_{\mathrm{BKW}}(\vec{T}_1) = -1$ and $\deg_{\mathrm{BKW}}(\vec{T}_2) = +1$)



Note that composing them with the "cups" at the bottom gives an element of $\mathrm{EXT}(\widehat{u}, \widehat{u})$ which is a number $\bar{\mathbb{Q}}$. Moreover, they are really "duals": From the four possibilities for composition, only two give non-zero numbers. In fact, one can loosely say that a combinatorial evaluation of closed $\mathfrak{sl}_n$-foams was already known by the higher Specht theory from Brundan, Kleshchev and Wang [8] or at least Hu and Mathas, see for example Theorem 6.17 in [30].

Remark 5.23. Using the *cell modules* (which can be constructed explicitly from the cellular basis, see Section 2 in [30]), we get two sets

$$\mathcal{D} = \{D^{\vec{\lambda}}\{k\} \mid \vec{\lambda} \in \tilde{\mathfrak{P}}^n_{c(\vec{k})},\, k \in \mathbb{Z}\} \text{ and } \mathcal{P} = \{P^{\vec{\lambda}}\{k\} \mid \vec{\lambda} \in \tilde{\mathfrak{P}}^n_{c(\vec{k})},\, k \in \mathbb{Z}\},$$

where $\tilde{\mathfrak{P}}^n_{c(\vec{k})} \subset \mathfrak{P}^n_{c(\vec{k})}$ is the subset of all n-multipartitions of $c(\vec{k})$ with $D^{\vec{\lambda}} \neq 0$. These form a complete set of pairwise non-isomorphic, graded, simple $H_n(\Lambda)$-modules and pairwise non-isomorphic, graded, projective indecomposable $H_n(\Lambda)$-modules respectively.

Furthermore, following the same approach as indicated in Remark 4.25 in [75], one can verify that these sets under the isomorphism of the (split) Grothendieck groups

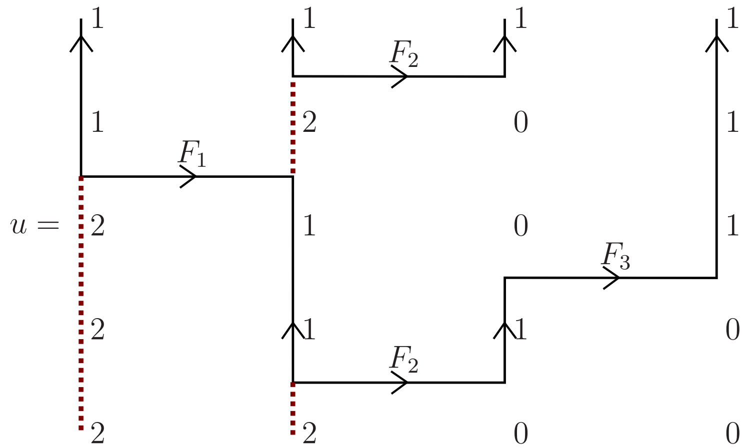
$$K_0^{(\oplus)}(\mathcal{W}_\Lambda^{(p)}) \otimes_{\mathbb{Z}[q,q^{-1}]} \bar{\mathbb{Q}}(q) \cong W_\Lambda = \bigoplus_{\vec{k} \in \Lambda(m,n\ell)_n} W_n^{(*)}(\vec{k})$$

correspond to the canonical and dual canonical basis respectively. Here the $\mathcal{W}_\Lambda^{(p)}$ are certain categories of modules over $H_n(\Lambda) \cong \check{R}(\Lambda)$, see Definition 7.1 in [50].

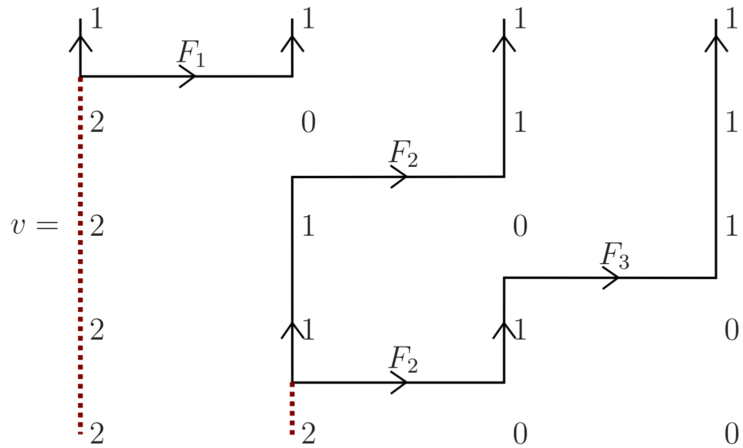
5.1.7. *An example.* We conclude this section with an example - we hope that it helps the reader.

Example 5.24. We will "cheat" a bit now in order to give a hopefully illustrating example how the graded cellular basis works. First let us fix $n = 2$, $\ell = 2$ and $\vec{k} = (1, 1, 1, 1)$, i.e. we will give a $\mathfrak{sl}_2$ example with $v_h = v_{(2^2)}$. We "cheat", because we do not use matrix factorizations in this example, but Bar-Natan's cobordisms [2] (not even Blanchet's cobordisms, i.e. everything below is only true up to a sign, see [5] and [45]). The reason is that the usage of these cobordisms illustrates without to many technical difficulties why the HM-basis really works so well. To cheat even more: We also ignore any shifts and gradings in this example.

We use the standard arc basis which in this case consists of the two $\mathfrak{sl}_2$-webs $u = F_2F_1F_3F_2v_{(2^2)}$



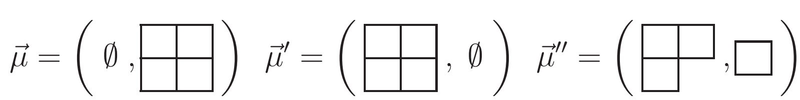
and $v = F_1F_2F_3F_2v_{(2^2)}$



In this case, the flows on these $\mathfrak{sl}_2$-webs are completely determined by the cut-line and we have six flows: The two canonical flows $S_c(u) = (\{2\},\{2\},\{1\},\{1\})$ and $S_c(v) = (\{2\},\{1\},\{2\},\{1\})$ and the two "anti-canonical" flows $S^c(u) = (\{1\},\{1\},\{2\},\{2\})$ and $S^c(v) = (\{1\},\{2\},\{1\},\{2\})$. Moreover, the $\mathfrak{sl}_n$-web v has two additional flows, namely $S_1(v) = (\{1\},\{2\},\{2\},\{1\})$ and $S_2(v) = (\{2\},\{1\},\{1\},\{2\})$.

What are we expecting to get now? A cellular basis should give rise to so-called *Specht modules* $\{S_i\}$, which can be *explicitly* obtained from the cellular basis. Moreover, these Specht modules should determine the *simple modules* $\{S_i/\mathrm{rad}(S_i)\}$. These (or more precise: their indecomposable projective covers $\{P_i\}$) will decategorify to the dual canonical basis. In the $\mathfrak{sl}_2$ case it is well-known that the arc basis is the dual canonical basis (see also Proposition 4.20), and thus, the projective modules P_u and P_v will be the corresponding indecomposable projective modules.

Thus, we expect two different "important" idempotents $e(\vec{\lambda})$ and $e(\vec{\mu})$, since these will determine the Specht modules. And we expect different dot placements $d(\cdot)$ for them, since both, idempotent and dot placement, depend *only* on the cut-line. And this is exactly what we get: We have six different 2-multipartitions (one for each flow at the boundary), namely (for $S_c(u)$, $S^c(u)$ and $S^c(v)$)



and (for $S_c(v), S_1(v)$ and $S_2(v)$ respectively)

$$\vec{\lambda} = \left(\ydiagram{1}\,, \ydiagram{2,1} \right) \quad \vec{\lambda}' = \left(\ydiagram{1,1}\,, \ydiagram{2} \right) \quad \vec{\lambda}'' = \left(\ydiagram{2}\,, \ydiagram{1,1} \right)$$

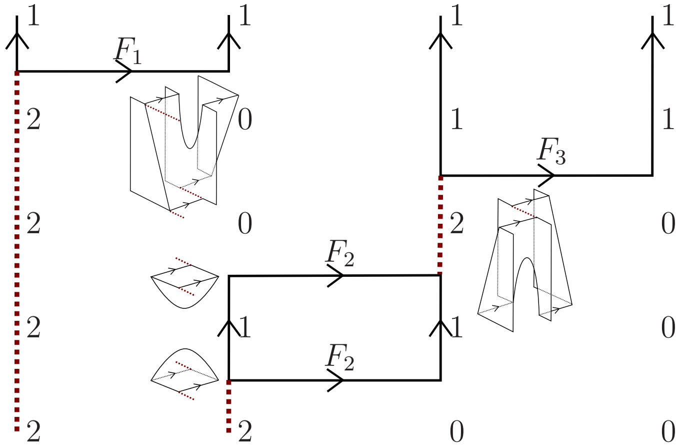
where $\vec{\mu}$, $\vec{\mu}'$ and $\vec{\mu}''$ have the same residue sequence $r(\vec{\mu}) = r(\vec{\mu}') = r(\vec{\mu}'') = (2,3,1,2)$ (recall the shift of the residue by $\ell = 2$ and one fills in numbers from left to right and top to bottom with rows first). Moreover, $r(\vec{\lambda}) = (2,2,3,1), r(\vec{\lambda}') = (2,1,2,3)$ and $r(\vec{\lambda}'') = (2,3,2,1)$.

Thus, we have $e(\vec{\mu}) = e(\vec{\mu}') = e(\vec{\mu}'') = \mathrm{id}_{F_2F_1F_3F_2v_{(2^2)}} \neq e(\vec{\lambda}) = \mathrm{id}_{F_1F_3F_2F_2v_{(2^2)}}$ and two additional $e(\vec{\lambda}') = \mathrm{id}_{F_1F_2F_1F_2v_{(2^2)}}$ and $e(\vec{\lambda}'') = \mathrm{id}_{F_1F_2F_3F_2v_{(2^2)}}$.

Moreover, since the dot placement is given by addable nodes to the right, we have no dots for $\vec{\mu}$, two dots for $\vec{\mu}'$ and one dot for the other four 2-multipartitions. The reader is invited to check that the Specht module for the $\vec{\mu}$'s, after modding out by the radical, is exactly the P_u. Moreover, we do not get too much: The elements for the two flows $S_1(v)$ and $S_2(v)$ will give rise to two nilpotent elements (with one dot each). Thus, they do not belong to the set $\tilde{\mathfrak{P}}^n_{c(\vec{k})}$ from Remark 5.23 since modding out by the radical will kill them (they are "unimportant").

We do the other in more details now, since it illustrates how the HM-basis does exactly what one would expect if one could guess the answer (as in this case), but works even if it is impossible to guess the answer (as in almost all other cases). The idempotent $e(\vec{\lambda})$ in this case is the id on

$w =$



The main question now can be seen as follows. The canonical flow on v works not only for v, but also for u (where it is a "mixed" flow). But since the dot placement and the idempotent is completely determined by the cut-line and one can not distinguish between the two just on the cut-line, the question is what is a "good" idempotent for $\vec{\lambda}$. The answer $e(\vec{\lambda}) = \mathrm{id}_{F_1F_3F_2F_2v_{(2^2)}}$, that is the identity on the $\mathfrak{sl}_2$-web above, can be seen as the "smallest common multiple" between u and v. That is, one can easily go from w to either u or v by using saddle moves $F_iF_{i\pm 1} \to F_{i\pm 1}F_i$ indicated above. We note that one has to use two saddles to go to u: First $F_3F_2 \to F_3F_2$ (bottom saddle above) and then $F_1F_2 \to F_2F_1$ (top saddle above), but only the bottom one to go to v.

The two possible extensions of $\vec{\lambda}$ are the canonical flow on v and the "mixed" on u given by

$$\vec{T}_c = \left(\begin{ytableau} 3 \end{ytableau}\,, \begin{ytableau} 1 & 2 \\ 4 \end{ytableau} \right) \quad \vec{T}_m = \left(\begin{ytableau} 4 \end{ytableau}\,, \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \right) \quad T_{\vec{\lambda}} = \left(\begin{ytableau} 1 \end{ytableau}\,, \begin{ytableau} 2 & 3 \\ 4 \end{ytableau} \right),$$

where the rightmost filling is the standard filling. Thus, in order to go from $T_{\vec{\lambda}}$ to the others, one has to use the permutations $\tau_1(2,2)\tau_2(3,2)\vec{T}_c = T_{\vec{\lambda}}$ in the first and $\tau_1(2,2)\tau_2(3,2)\tau_3(1,2)\vec{T}_m = T_{\vec{\lambda}}$

in the second case. The $\tau_k(i,j)$ correspond to a *cup-cap-move* (if $i=j$, see in the figure above), a *saddle* (if $|i-j|=1$) or a shift (if $|i-j|>1$). Thus, if we use $\sigma=\tau_1(2,2)\tau_2(3,2)$ and $\tilde{\sigma}=\tau_1(2,2)\tau_2(3,2)\tau_3(1,2)$ as shorthand notations, we see that the four elements

$$v^*v \rightsquigarrow \sigma^* e(\vec{\lambda})d(\vec{\lambda})\sigma \quad u^*v \rightsquigarrow \tilde{\sigma}^* e(\vec{\lambda})d(\vec{\lambda})\sigma \quad v^*u \rightsquigarrow \sigma^* e(\vec{\lambda})d(\vec{\lambda})\tilde{\sigma} \quad u^*u \rightsquigarrow \tilde{\sigma}^* e(\vec{\lambda})d(\vec{\lambda})\tilde{\sigma}$$

(here $d(\vec{\lambda})$ denotes a dot on a cylinder between the internal circle) which correspond to the four possible combinations v^*v, u^*v, v^*u and u^*u, gives exactly the answer one would expect.

That is, all of them remove the internal circle by closing the dotted cylinder using a cap at the top and a cup at the bottom (with the Bar-Natan relations: This is a dotted sphere and hence equals 1). Now the first one for example uses the saddle move given by $\tau_2(3,2)$ to connect the internal circle to one of the boundary sheets and the end result is just two un-dotted sheets (as one would guess). The reader is invited to draw the pictures for the other three possibilities. Note that in the last case the algorithm creates a "neck" (in the language of Bar-Natan's cobordism) that one can cut giving a linear combination in contrast to the case for the "anti-canonical" which gives two dotted cylinders. Thus, they are all nilpotent except $\sigma^* e(\vec{\lambda})d(\vec{\lambda})\sigma$.

5.2. **Connections to the Khovanov-Rozansky $\mathfrak{sl}_n$-link homologies.**

5.2.1. *Short overview.* Let us shortly sketch the structure of this section which can be seen as a categorification of the results of Section 4.2. We assume that the (oriented) link diagrams L_D are in a *Morse position* such that all crossings and point *upwards*. Moreover, as in Section 4.2, all crossings and have the colors $a,b\in\{0,\dots,n\}$ where we *always* assume that the strand going from bottom left to top right has the color a (compare to our discussion of the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-polynomial in Definition 4.26). We point out that n will be fixed. We also use braid diagrams that we denote by B_D.

We assume that the reader is familiar with the classical construction of *Khovanov homology using cubes* (see e.g. [2]). Moreover, although it follows from our construction that one *does not need* $\mathfrak{sl}_n$-matrix factorizations (or $\mathfrak{sl}_n$-foams), we implicitly assume that the reader is familiar with colored Khovanov-Rozansky $\mathfrak{sl}_n$-link homology that can be found in [43] (uncolored) or [79] or alternatively [81]. We denote it by $\mathrm{KR}(L_D)^n$.

We want to use our translation from $\mathfrak{sl}_n$-webs to n-multitableaux and the resulting isomorphism from Theorem 5.16. The idea can be summarized as follows. Assume that a link diagram L_D is given as a sequence as in Lemma 4.30. We can then define a cube where we put at each vertex a string of $F_i^{(j)}$'s jumping from a highest (n^ℓ) to a lowest weight. We associate to each (n^ℓ) a *canonical sequence of $F_i^{(j)}$'s*, denoted by $F^c_{(n^\ell)}$, only made of *leash shifts* (we can see this as an empty diagram). Then, at each vertex, there is a module over $\check{R}(\Lambda)$ given by $\hom_{\check{R}}(F^c_{(n^\ell)},\cdot)$ (shorthand notation for $\hom_{\check{R}(\Lambda)}$) and the differentials are $\check{R}(\Lambda)$-module maps. Using Theorem 5.16 we see that these are modules over the $\mathfrak{sl}_n$-web algebra $H_n(\Lambda)$. Thus, we see that our purely combinatorial homology agrees with the colored Khovanov-Rozansky $\mathfrak{sl}_n$-link homology.

For the computation: We can therefore use the thick HM-basis from Definition 5.10 for the sources of the differential and the dual basis from Remark 5.21 for the targets. We get this way an element $\hom_{\check{R}}(F^c_{(n^\ell)},F^c_{(n^\ell)})$. Since this hom-space is *one dimensional*, we get numbers in $\bar{\mathbb{Q}}$ which give the entries of the matrices for the differentials.

The structure of this section is essentially the same as in Section 4.2. That is, we start by recalling Chuang and Rouquier's Rickard complex from [23] (or rather a slightly arranged form

of Cautis version from [15] of it) and then our F-braiding complex in Definition 5.28. We use the Rickard complex to show that our definition respects the braid moves, see Proposition 5.32, which gives us the possibility to use local simplification as we sketch in Remark 5.33. Then we define the homology in Definition 5.36 and show that it agrees with the colored Khovanov-Rozansky $\mathfrak{sl}_n$-link homology in Theorem 5.37. Afterwards we give our purely combinatorial way to calculate the homology in Subsection 5.2.5. As usual, we have included lots of examples to (hopefully) help the reader.

Remark 5.25. We will formulate everything in this section in a mixture of different notations. First we note that we freely switch between the notions $\mathfrak{sl}_n$-webs, their associated matrix factorizations, string of $F_i^{(j)}$'s and string of $\mathcal{F}_i^{(j)}$'s. We hope that is not too confusing.

Moreover, while we talk about braids, we stay in the KL-R part of $\check{\mathcal{U}}(\mathfrak{sl}_m)$ and only go to the cyclotomic quotient for the $\mathfrak{sl}_n$-link homologies. The reason is that we can not formulate the complex *locally* in the thick cyclotomic KL-R, because, in our convention, we would have to start at a weight (n^ℓ) for some ℓ. We try to distinguish them as follows: The pictures for the KL-R part of $\check{\mathcal{U}}(\mathfrak{sl}_m)$ have orientations (in our notation they are oriented downwards) and the ones for $\check{R}_\Lambda$ do not have orientations. We use for the 2-Schur quotient $\check{\mathcal{S}}(m, n\ell)_n$ (see below before Lemma 5.30) of $\check{\mathcal{U}}(\mathfrak{sl}_m)$ the same notations as for $\check{\mathcal{U}}(\mathfrak{sl}_m)$ itself.

5.2.2. *The Rickard complex.* Recall that Chuang and Rouquier's Rickard complex from [23] can be seen as a *categorification* of the quantum Weyl group action on V_N from 2.1.1 that acts by a reflection isomorphism between the k-th and $-k$-th weight space. We present a slightly adopted version of Cautis's presentation from [15] here. The reason is that this fits to the convention for the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial given in Definition 4.26. Moreover, recall that we use $\mathfrak{gl}_n$-weights $\vec{k}$ in our pictures. The definition of the Rickard complex given in [15] on the other hand uses $\mathfrak{sl}_n$-weights. Thus, we convert them by using Equation 3.2.8 where we (as usual) denote the weights by the same symbol.

The reader familiar with [45] or [61] should be careful since our conventions are slightly different. Moreover, we denote by $\mathcal{T}v_{\vec{k}}$ usually the F-braiding complex given in Definition 5.28 and the Rickard version by $\mathcal{T}\mathbf{1}_{\vec{k}}$.

Given a suitable 2-category $\mathfrak{C}$, recall that the 2-category $\mathbf{Kom}_{\mathrm{gr}}(\mathfrak{C})$ has the same objects as $\mathfrak{C}$, but the morphisms are complexes of $\mathfrak{C}$ and the 2-morphisms are chain maps between these complexes. Moreover, everything should be graded and morphisms should preserve the degree.

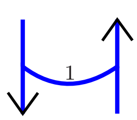
Definition 5.26. Given a $\vec{k}$ with a, b in the i-th and $i+1$-th entry, we define the *i-th positive Rickard complex $\mathcal{T}_i^+\mathbf{1}_{\vec{k}}$* in $\mathbf{Kom}_{\mathrm{gr}}(\check{\mathcal{U}}(\mathfrak{sl}_m))$ as

$$\mathcal{F}_i^{(a-b)}\mathbf{1}_{\vec{k}}\{q_0\} \xrightarrow{d_0^{\mathrm{R}}} \mathcal{F}_i^{(a+1-b)}\mathcal{E}_i\mathbf{1}_{\vec{k}}\{q_1\} \xrightarrow{d_1^{\mathrm{R}}} \mathcal{F}_i^{(a+2-b)}\mathcal{E}_i^{(2)}\mathbf{1}_{\vec{k}}\{q_2\} \xrightarrow{d_2^{\mathrm{R}}} \dots$$

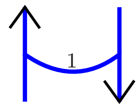
for $b \leq a$ with shifts $q_k = -b + k$ and

$$\mathcal{E}_i^{(-a+b)}\mathbf{1}_{\vec{k}}\{q_0\} \xrightarrow{d_0^{\mathrm{R}}} \mathcal{E}_i^{(-a+1+b)}\mathcal{F}_i\mathbf{1}_{\vec{k}}\{q_1\} \xrightarrow{d_1^{\mathrm{R}}} \mathcal{E}_i^{(-a+2+b)}\mathcal{F}_i^{(2)}\mathbf{1}_{\vec{k}}\{q_2\} \xrightarrow{d_2^{\mathrm{R}}} \dots$$

for $a < b$ with shifts $q_k = -a + k$. In both cases the leftmost part is in homology degree zero. The differentials are given by

$d_k^{\mathrm{R}} =$  $: \mathcal{F}_i^{(a+k-b)}\mathcal{E}_i^{(k)}\mathbf{1}_{\vec{k}}\{q_k\} \to \mathcal{F}_i^{(a+k+1-b)}\mathcal{E}_i^{(k+1)}\mathbf{1}_{\vec{k}}\{q_{k+1}\}$

and

$$d_k^{\mathrm{R}} = $$  $$: \mathcal{E}_i^{(-a+k+b)}\mathcal{F}_i^{(k)}\mathbf{1}_{\vec{k}}\{q_k\} \to \mathcal{E}_i^{(-a+k+1+b)}\mathcal{F}_i^{(k+1)}\mathbf{1}_{\vec{k}}\{q_{k+1}\}$$

for the two cases respectively. They are both invertible up to homotopy and we denote their inverses (that should correspond to our negative crossings) by $\mathbf{1}_{\vec{k}}\mathcal{T}_i^-$ and call them *i-th negative Rickard complex $\mathbf{1}_{\vec{k}}\mathcal{T}_i^-$*. They are also in $\mathbf{Kom}_{\mathrm{gr}}(\check{\mathcal{U}}(\mathfrak{sl}_m))$.

As an example for $\vec{k} = (1,1)$ we have

$$\mathcal{T}_1^+\mathbf{1}_{\vec{k}} = \mathbf{1}_{\vec{k}}\{-1\} \longrightarrow \mathcal{F}_i\mathcal{E}_i\mathbf{1}_{\vec{k}}\{0\}$$

which is essentially categorification of the Kauffman bracket (the reader is encouraged to draw the pictures for the corresponding $\mathfrak{sl}_n$-webs).

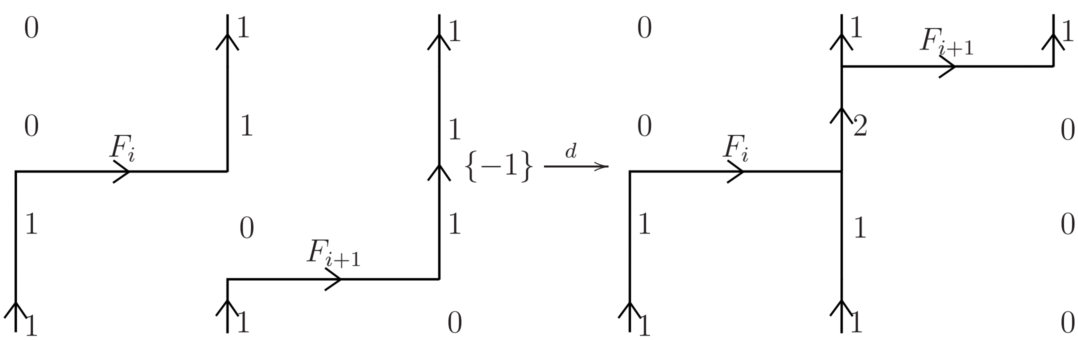
The following is Theorem 6.3 in Cautis and Kamnitzer's paper [16] and highly non-trivial.

Theorem 5.27. *Given an integrable 2-representation $\psi\colon \check{\mathcal{U}}(\mathfrak{sl}_m) \to \mathfrak{K}$, then the images under ψ of the Rickard complexes satisfy the braid relations in $\mathbf{Kom}_{\mathrm{gr}}(\mathfrak{K})$ up to homotopy.* □

5.2.3. *The F-braiding complex.* In this subsection we define the categorification of the (colored) braiding operator $T_{a,b,i}^k$ from Definition 4.28. We call the categorification the *(colored) F-braiding complex*. We start with the "un"-colored case where we still draw the pictures. For the colored case we *do not draw* the $\mathfrak{sl}_n$-webs anymore but use our F notation instead.

Definition 5.28. (**Braiding complex - only F's**) Recall that we defined in Definition 4.28 the braiding operators T_i^k for $k = 0, 1$ which acts on a weight $\vec{k}$ with i and $i+1$ entry equal to 1 and the $i+2$-th entry equal to zero. The *F-braiding complex $\mathcal{T}_i^+ v_{\vec{k}}$* is then defined to be

$$\mathcal{T}_i^+ v_{\vec{k}} =$$



with differential $d =$: $F_iF_{i+1}v_{\vec{k}} \to F_{i+1}F_iv_{\vec{k}}$ and leftmost component in homology degree zero. The *braiding complex $\mathcal{T}_i^- v_{\vec{k}}$* is defined in the same way, but with switched pictures, rightmost component in homology degree zero, a differential $d =$: $F_{i+1}F_iv_{\vec{k}} \to F_iF_{i+1}v_{\vec{k}}$ and a q-degree shift by 1 for the rightmost component. In an algebraic notation this will be

$$\mathcal{T}_i^- v_{\vec{k}} = 0 \longrightarrow T_i^0 v_{\vec{k}} = F_{i+1}F_iv_{\vec{k}} \xrightarrow{d} T_i^1 v_{\vec{k}} = F_iF_{i+1}v_{\vec{k}}\{1\} \longrightarrow 0.$$

We encourage the reader to draw the pictures.

Now assume that $\vec{k}$ has a in the i-th and b in the $i+1$-th entry and the $i+2$-th entry equal to zero. The colored *positive F-braiding complex* $\mathcal{T}^{+}_{a,b,i}v_{\vec{k}}$ is then defined to be

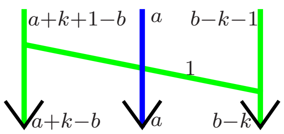
$$F^{(a-b)}_{i+1}F^{(a)}_{i}F^{(b)}_{i+1}v_{\vec{k}}\{q_0\}\xrightarrow{d_0}F^{(a+1-b)}_{i+1}F^{(a)}_{i}F^{(b-1)}_{i+1}v_{\vec{k}}\{q_1\}\xrightarrow{d_1}\dots\xrightarrow{d_{b-1}}F^{(a)}_{i+1}F^{(a)}_{i}F^{(0)}_{i+1}v_{\vec{k}}\{q_b\}$$

in the case $b\leq a$ and for $a<b$ we use

$$F^{(a)}_{i}F^{(a)}_{i+1}F^{(0)}_{i}\{q_0\}\xrightarrow{d_0}F^{(a-1)}_{i}F^{(a)}_{i+1}F^{(1)}_{i}v_{\vec{k}}\{q_1\}\xrightarrow{d_1}\dots\xrightarrow{d_{b-1}}F^{(0)}_{i}F^{(a)}_{i+1}F^{(a)}_{i}v_{\vec{k}}\{q_a\}$$

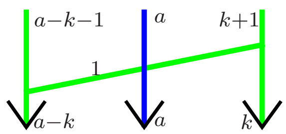
with the leftmost term in homology degree zero. The q-degree shifts are $q_k=-b+k$ in the first and $q_k=-a+k$ in the second case (compare to Definition 4.28).

The differentials are given by (the thickness of the middle edge is 1)

$d_k =$ 

$: F^{(a+k-b)}_{i+1}F^{(a)}_{i}F^{(b-k)}_{i+1}v_{\vec{k}}\{q_k\}\to F^{(a+k+1-b)}_{i+1}F^{(a)}_{i}F^{(b-k-1)}_{i+1}v_{\vec{k}}\{q_{k+1}\}$

in the case $b\leq a$ and by (the thickness of the middle edge is 1)

$d_k =$ 

$: F^{(a-k)}_{i}F^{(a)}_{i+1}F^{(k)}_{i}v_{\vec{k}}\{q_k\}\to F^{(a-k-1)}_{i}F^{(a)}_{i+1}F^{(k+1)}_{i}v_{\vec{k}}\{q_{k+1}\}$

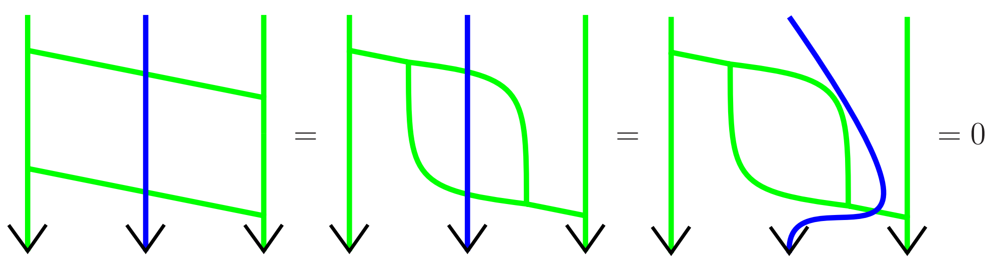
in the case $a<b$. We note that the special case $a=b=1$ is the usual KL-R crossing from above.

The colored *negative F-braiding complex* $\mathcal{T}^{-}_{a,b,i}v_{\vec{k}}$ is defined by turning "everything around": Switched pictures, rightmost component in homology degree zero, the differentials are reflections of the ones from before and q-degree shifts $q_k=b-k$ in the $b\leq a$ and $q_k=a-k$ in the $a<b$ case. The reader is encouraged to write down the complexes.

Since the a,b are encoded by $v_{\vec{k}}$ we tend not to write the a and b explicitly.

Lemma 5.29. *The F-braiding complex $\mathcal{T}^{\pm}_{a,b,i}v_{\vec{k}}$ is an element of* $\textbf{\textit{Kom}}_{\mathrm{gr}}(\mathcal{U}(\mathfrak{sl}_m))$, *i.e. the differentials preserve the degree and $d_{k+1}\circ d_k=0$.*

Proof. Let us skip the labels in the following. We have in the positive $b\leq a$ case



where the first equation follows from the associativity of splitters and merges (see e.g. Proposition 2.2.4 in [42]), the second from the pitchfork relation (see e.g. Proposition 4 in [70]) and the third is a direct consequence of the definition of splitters and merges (see e.g. Equation 2.64 in [42]). We leave the positive $a<b$ case and the negative cases to the reader.

The difference between two shifts is $q_k-q_{k+1}=-1$. Thus, the differentials have to be of degree 1 in order to be degree preserving. Recall that slitters and merges are of degree $-jj'$ (if they split $j+j'$ into j and j' or vice versa for merges). Since the middle edges are of thickness 1, we can

read of minus the degree of them by looking at the bottom left and top right boundary in the $b \leq a$ case and at the bottom right and top left boundary in the $a < b$ case. For both the sum is $a-1$. Thus, since the thick middle crossing is of degree a, the differentials are of degree 1. We leave the negative cases to the reader again. □

I thank Queffelec and Rose that they pointed out that using the Rickard complex $\mathcal{T}_i^{+}\mathbf{1}_{\vec{k}}$ is essentially equivalent to the F-braiding complex $\mathcal{T}_i^{+}v_{\vec{k}}$. Part (a) can be seen as a categorification of Lemma 4.9. For analogous statements see Lemma 3.13 and Remark 3.14 in [61].

Before we start recall that the q-Schur 2-algebra $\mathcal{S}(m,n\ell)_n$ is obtained from $\mathcal{U}(\mathfrak{gl}_n)$ by taking the quotient by setting all 2-morphisms that have a region with a label not in $\Lambda(m,n\ell)_n$ to zero. For details see [52]. The reader may convince herself/himself that it is in fact not a big deal to define $\check{\mathcal{S}}(m,n\ell)_n$ that we will use in the following and denote just by $\check{\mathcal{S}}$.

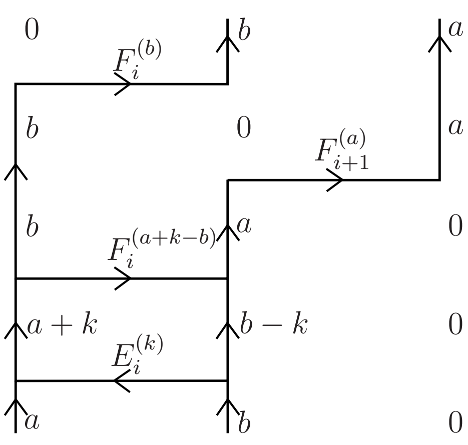
Lemma 5.30. *Denote by* $\textbf{\textit{Kom}}^h_{\mathrm{gr}}(\check{\mathcal{S}})$ *the homotopy category of complexes for* $\check{\mathcal{S}}(m,n\ell)_n$ *and sufficiently large* m.

(a) *Let* $u,v \in W_n(\vec{k})$ *be two isotopic* $\mathfrak{sl}_n$*-webs with a possible different presentation under* q*-skew Howe duality* $u = \mathrm{qH}v_{(n^\ell)}$ *and* $v = \mathrm{qH}'v_{(n^\ell)}$ *(here* qH *and* qH' *consists of strings of* $E_i^{(j)}$*'s and* $F_i^{(j)}$*'s). Then there exists an isomorphism in* $\check{\mathcal{U}}(\mathfrak{sl}_m)$ *between the corresponding* $\mathcal{E}_i^{(j)}$*'s and* $\mathcal{F}_i^{(j)}$*'s realizing this isotopy. Moreover, all* $\mathfrak{sl}_n$*-web isotopies come already from isomorphisms in the KL-R part of* $\check{\mathcal{U}}(\mathfrak{sl}_m)$ *for sufficiently large* m.

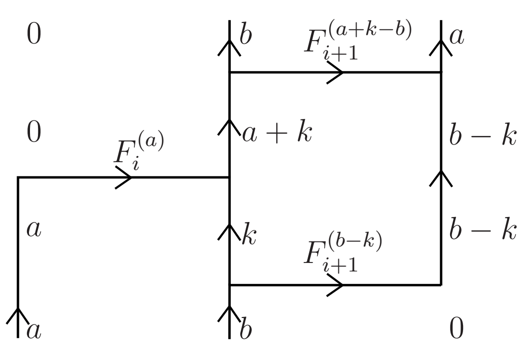
(b) *The Rickard complex* $\mathcal{F}_i^{(b)}\mathcal{F}_{i+1}^{(a)}\mathcal{T}_i^{+}\mathbf{1}_{\vec{k}}$ *is the same as* $\mathcal{T}_i^{+}v_{\vec{k}}$ *in* $\textbf{\textit{Kom}}^h_{\mathrm{gr}}(\check{\mathcal{S}})$ *in the case* $b \leq a$. *Analogous statements are true for the other cases.*

Proof. (a) This is just a consequence of the results from the previous sections. To be more precise, by Lemma 4.9 and Proposition 4.8 we see that each $\mathfrak{sl}_n$-web corresponds to an equivalence class of n-multipartitions (taking isotopies in account). By Theorem 5.16 and Mackaay's Corollary 7.6 in [50] (that the split Grothendieck group of $\mathcal{W}^p_\Lambda$ is equivalent to the $\mathfrak{sl}_n$-web space $W_n(\Lambda)$) we see that all $\mathfrak{sl}_n$-web isotopies, if only $F_i^{(j)}$'s are involved, have to come from a certain $\check{R}(\Lambda)$. If $E_i^{(j)}$'s are involved, then the $\mathfrak{sl}_n$-webs still give the same on the level of Grothendieck groups, but the isotopies come from $\check{\mathcal{U}}(\mathfrak{sl}_m)$ for a suitable m (rewriting E's in terms of F's increases the m).

(b) We note that any isomorphism is not sufficient, since it has to give rise to a chain map. We therefore give such isomorphisms below which come from the following isomorphisms between the $\mathfrak{sl}_n$-webs



and



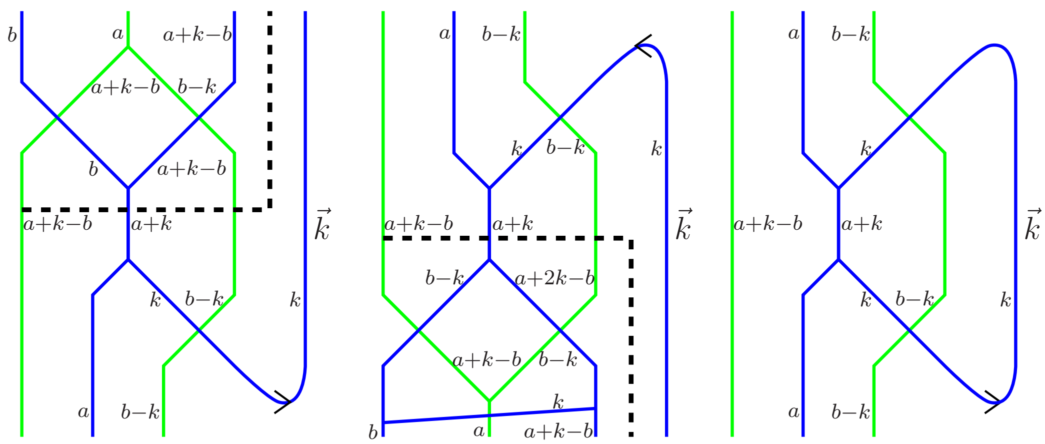
where the first $\mathfrak{sl}_n$-web is for the Rickard complex (which categorifies the rules from Definition 4.26) and the second is for the F-braiding complex (which, on the other hand, categorifies the rules from Definition 4.28).

We do not care for signs here because, if some signs for some squares as below do not work, then we can change them by multiplying with an extra sign for the right arrow for the corresponding square (starting at the leftmost). Moreover, we note that using $\check{\mathcal{S}}$ ensures that the complexes are all bounded from left and right. Thus, the sign change procedure is well-defined and terminates.

We now consider the following square where the k-th part of the Rickard complex is the top left and the k-th part of the F-braiding complex is the bottom left (with $\vec{k} = (,\dots,a,b,0,\dots)$).

$$\begin{tikzcd}
\mathcal{F}_i^{(b)}\mathcal{F}_{i+1}^{(a)}\mathcal{F}_i^{(a+k-b)}\mathcal{E}_i^{(k)}\mathbf{1}_{\vec{k}}\{q_k\} \arrow[r, "d_k^{\mathrm{R}}"] \arrow[d, shift left, "g_k"] & \mathcal{F}_i^{(b)}\mathcal{F}_{i+1}^{(a)}\mathcal{F}_i^{(a+k+1-b)}\mathcal{E}_i^{(k+1)}\mathbf{1}_{\vec{k}}\{q_{k+1}\} \arrow[d, shift left, "g_{k+1}"] \\
F_{i+1}^{(a+k-b)}F_i^{(a)}F_{i+1}^{(b-k)}v_{\vec{k}}\{q_k\} \arrow[r, "d_k"] \arrow[u, shift left, "f_k"] & F_{i+1}^{(a+k+1-b)}F_i^{(a)}F_{i+1}^{(b-k-1)}v_{\vec{k}}\{q_{k+1}\} \arrow[u, shift left, "f_{k+1}"]
\end{tikzcd}$$

The maps f_k (left) and g_k (middle) are given by



We have also indicated the thickness of the strands in order to help the reader. We note that part of these 2-morphisms (the ones that we have separated) are exactly the same 2-morphisms as in Section 4.2 of [69]. The partition $\alpha \in P(0,k)$ in Stošić's notation there will be empty. Note that already the marked parts are of degree zero.

The proof that $g_k \circ f_k = \pm\mathrm{id}_{F_{i+1}^{(a+k-b)}F_i^{(a)}F_{i+1}^{(b-k)}v_{\vec{k}}}$ and $f_k \circ g_k = \pm\mathrm{id}_{\mathcal{F}_i^{(b)}\mathcal{F}_{i+1}^{(a)}\mathcal{F}_i^{(a+k-b)}\mathcal{E}_i^{(k)}\mathbf{1}_{\vec{k}}}$ follows from calculations of Stošić in [69]. For example, to see the first identity, one can use the equation in the proof of Lemma 4 in [69] (recall that we ignore signs). This reduces the diagram to the right picture above. Then one can use the "Opening of a thick edge" (Proposition 5 in [69]) followed by the "Thick R3 move" (Proposition 7 in [69]) and apply "Higher reduction of bubbles" (Proposition 5.2.9 in [42]) to see that this is just the identity (up to a sign). I order to keep the length of this paper in reasonable boundaries (ok, we totally failed), we do not so the calculations and leave them to the reader. We note that most of the summands that come from the relations cited above will collapse due to weight reasons. Moreover, we leave it to the reader to verify that these f_k, g_k make the square commutative (up to a sign). Again, this only works in the Schur quotient $\check{\mathcal{S}}$. The other cases are again similar in the sense that they can be deduced from $\mathfrak{sl}_n$-web isotopies (and in the sense that they need non-trivial calculations) and left to the reader. This shows (b). $\square$

Definition 5.31. **(Khovanov-Rozansky $\mathfrak{sl}_n$-braid complex only using F's)** Given an oriented, colored braid diagram B_D with cr crossings and a fixed presentation of it using q-skew Howe duality

$$B_D = \prod_k \tilde{F}_{i_k}^{(j_k)} v_{(n^\ell)}, \text{ with } \tilde{F}_{i_k}^{(j_k)} \text{ as in } 4.30,$$

with $T^{\pm}$'s for the or , we assign to it the $\mathfrak{sl}_n$*-braid complex via F's* by

$$[\![B_D]\!]_F^n = \prod_k F_{i_k}^{(j_k)} \cdot \bigotimes_{k=1}^{\mathrm{cr}} \mathcal{T}_{i_k}^{\pm} \cdot \prod_k F_{i_k}^{(j_k)} v_{(n^\ell)},$$

where we allow $F_i^{(j)}$'s between the $T_{i_k}^{\pm}$ if they appear in the fixed presentation above between them. Moreover, the weights $\vec{k}$ for the $\mathcal{T}$'s from Definition 5.28 have to be suitably rearranged and the corresponding diagrams are the identities on the components $\prod_k F_{i_k}^{(j_k)}$.

Proposition 5.32. *The complex $[\![B_D]\!]_F^n$, viewed in the corresponding homotopy category of complexes $\textbf{\textit{Kom}}_{\mathrm{gr}}^h(\check{\mathcal{S}})$, gives an invariant of framed braids. That is, it does not depend the braid moves.*

Proof. This is a consequence of Theorem 5.27 combined with Lemma 5.30 and the easy to deduce fact that $\check{\mathcal{S}}$ is an integrable 2-representation $\psi\colon \check{\mathcal{U}}(\mathfrak{sl}_m) \to \check{\mathcal{S}}$. $\square$

Remark 5.33. We point out that there is way to prove Proposition 5.32 directly in our framework and extend it to "braid-like tangles". Latter are more flexible then braids and satisfy additional moves called *tangle braid moves* (see e.g. Figure 2 in [15] or Lemma X.3.5 in [32]).

This alternative proof is based on the higher quantum Serre relations and their categorification given in [69]. Moreover, we think that these complexes can be used for a "divide and conquer" strategy for computations á la Bar-Natan [1]. But this paper is already long enough (or worse: Too long!) so we only sketch how it should work. Compare also to our proof of Theorem 4.31.

Given the set-up as in the proof of Theorem 4.31, we get a complex (recall that $v = v_{\dots,1,1,0,0,\dots}$)

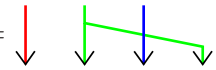
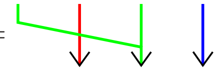
$$\begin{tikzcd}
 & F_{i+1}F_{i+2}F_iF_{i+1}v\{0\} \arrow[dr, ": F_iF_{i+1}\to F_{i+1}F_i"] & \\
F_{i+2}F_{i+1}F_iF_{i+1}v\{-1\} \arrow[ur, ": F_{i+2}F_{i+1}\to F_{i+1}F_{i+2}"] \arrow[dr, ": F_iF_{i+1}\to F_{i+1}F_i"'] & \oplus & F_{i+1}F_{i+2}F_{i+1}F_iv\{+1\} \\
 & F_{i+2}\textcolor{red}{F_{i+1}}\textcolor{red}{F_{i+1}}F_iv\{0\} \arrow[ur, "-: F_{i+2}F_{i+1}\to F_{i+1}F_{i+2}"'] &
\end{tikzcd}$$

There is an explicit isomorphism $\mathcal{F}_{i+1}\mathcal{F}_{i+1} \cong \mathcal{F}_{i+1}^{(2)}\{-1\} \oplus \mathcal{F}_{i+1}^{(2)}\{+1\}$ in $\check{\mathcal{U}}(\mathfrak{sl}_m)$, see Theorem 5.1.1 in [42] (the same is true in $\check{\mathcal{S}}$). This, in the $n = 2$ case, is just Bar-Natan's delooping from Lemma 3.1 in [1].

We get from this, focussing on the bottom path of the complex above, the following complex.

$$\begin{tikzcd}
F_{i+2}F_{i+1}F_iF_{i+1}v\{-1\} \arrow[r, "d_1"] & F_{i+2}F_{i+1}^{(2)}F_iv\{-1\} & \\
 & \oplus & \\
 & F_{i+2}F_{i+1}^{(2)}F_iv\{+1\} \arrow[r, "d_2"] & F_{i+1}F_{i+2}F_{i+1}F_iv\{+1\}.
\end{tikzcd}$$

The differentials will change as usual using Gauss elimination, see e.g. Lemma 3.2 in [1], to

$d_1 =$  and $d_2 =$ 

Now comes the clue: The top line is part of the null-homotopic complex defined in Theorem 7 in [69] (or a variant of it by exchanging E's to F's and indices) for $a = 2, b = 1$. We note that, due to weight reasons, most terms of Stošić's complex will be zero. On the other hand, the bottom line is part of the null-homotopic complex defined in Theorem 6 in [69] (again slightly re-arranged).

As explained in [1], the complex will collapse and the starting complex is homotopic to the trivial complex which shows the invariance under the second Reidemeister move.

5.2.4. *Colored $\mathfrak{sl}_n$-link homology using F's.* We are now ready to define our version of the colored Khovanov-Rozansky $\mathfrak{sl}_n$-link homology.

Definition 5.34. Given a weight (n^ℓ), we associate to it a *canonical sequence of $F_i^{(j)}$'s*, denoted by $F^c_{(n^\ell)}$, by applying iteratively $F_i^{(n)}$ to shift all n's to the right by shifting always the rightmost pair of the form $(\ldots, n, 0, \ldots)$ to $(\ldots, 0, n, \ldots)$.

Example 5.35. The canonical sequence associated to $(3,3,0,0)$ is $F_2^{(3)}F_1^{(3)}F_3^{(3)}F_2^{(3)}$. Another example is given in Example 5.22.

Definition 5.36. (**Khovanov-Rozansky $\mathfrak{sl}_n$-link homology only using F's**) Given a oriented, colored link diagram L_D with cr crossings $c_{a,b}$ and a fixed presentation of it using q-skew Howe duality

$$L_D = \prod_k F_{i_k}^{(j_k)} \cdot T^{\pm}_{i_{\mathrm{cr}}} \cdots T^{\pm}_{i_1} \cdot \prod_k F_{i_k}^{(j_k)} v_{(n^\ell)}$$

with $T^{\pm}$'s for the  or  (as before, we allow extra F's between the different $T^{\pm}$'s), we assign to it the *colored Khovanov-Rozansky $\mathfrak{sl}_n$-link homology via F's* by

$$[\![L_D]\!]^n_F = \hom_{\check{R}}(F^c_{(n^\ell)}, \prod_k F_{i_k}^{(j_k)} \cdot \bigotimes_{k=1}^{\mathrm{cr}} \mathcal{T}^{\pm}_{i_k} \cdot \prod_k F_{i_k}^{(j_k)} v_{(n^\ell)})$$

(we write shortly $\hom_{\check{R}}$ for $\hom_{\check{R}(\Lambda)}$) and

$$\mathrm{KR}(L_D)^n_{\mathrm{F}} = [\![L_D]\!]^n_{\mathrm{F}}\{\mathrm{power}(q)\}$$

where the shift in the q-degree $\{\mathrm{power}(q)\}$ is the same as power of the q in the product from 4.2.1. Moreover, the weights $\vec{k}$ for the $\mathcal{T}$'s from Definition 5.28 have to be suitably rearranged for the tensor product to make sense.

Theorem 5.37. *The complex $\mathrm{KR}(L_D)^n_{\mathrm{F}}$ is the same as $\mathrm{KR}(L_D)^n$ viewed as objects in the homotopy category of complexes **Kom**$^h_{\mathrm{gr}}(\mathcal{W}^p_\Lambda)$. Thus, it is an invariant of colored links and therefore invariant under the three Reidemeister moves and isotopies.*

A similar result can be concluded for the complex $[\![L_D]\!]^n_{\mathrm{F}}$, but one has to be very careful with possible degree shifts. We do not do it here. Moreover, we would like to prove the invariance directly in our set-up.

Proof. One part of the argument is very similar to the one used by Lauda, Queffelec and Rose to proof that their complex agrees with the Khovanov-Rozansky $\mathfrak{sl}_n$-link homology (for $n = 2, 3$), see Proposition 4.3 in [45]. One part of their argument is that the differentials in their complex

are, up to a sign, the same for both complexes. Then they use an argument similar to [60]. A very similar argument works for the complex $\mathrm{KR}(L_D)^n_{\mathrm{F}}$. Thus, we can ignore these signs in the following.

The rest is also easy to verify with our results from the previous sections. To be more precise, using Theorem 5.16, we see that our modules $\hom_{\check{R}}(F^c_{(n^\ell)}, \cdot)$ are graded isomorphic to modules over the $\mathfrak{sl}_n$-web algebra $H_n(\Lambda)$ defined by Mackaay. Thus, they are certain EXT-spaces of matrix factorizations associated to the underlying $\mathfrak{sl}_n$-webs (that we obtain from the string of $F_i^{(j)}$'s via the translation from Section 4.1).

Checking the definition of the differentials for $\mathrm{KR}(L_D)^n$ (that can be found in Section 7 in [43] or in the colored case in Definition 12.4 in [79] or alternatively in Section 5 and 6 in [81]) we see that they all can be obtained by applying the extended 2-functor from Theorem 3.36 to the Rickard complex from 5.26.

Now comes an important point that we like to proof in our setting directly. Using the isotopy invariance of $\mathrm{KR}(L_D)^n$ (see Theorem 2 in [43] or in the colored case see Theorem 1.1 in [79] or alternatively Theorem 1.3 in [81]) together with Lemma 5.30 we see that this induces a homotopy between $\mathrm{KR}(L_D)^n_{\mathrm{F}}$ and $\mathrm{KR}(L_D)^n$ which shows the first statement.

Since $\mathrm{KR}(L_D)^n$ is invariant under the Reidemeister moves, the same holds for $\mathrm{KR}(L_D)^n_{\mathrm{F}}$ as well. Thus, this finishes the proof. □

And $\mathrm{KR}(L_D)^n_{\mathrm{F}}$ categorifies the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial RT_n.

Corollary 5.38. *Given a oriented, colored link diagram L_D. Then the graded Euler characteristic of the complex $\mathrm{KR}(L_D)^n_{\mathrm{F}}$ gives $\mathrm{RT}_n(L_D)$.*

Proof. This is just a combination of Theorem 5.37 and e.g. Theorem 1.3 in [79]. □

Remark 5.39. An analogue of Definition 5.36 and Theorem 5.37 can be formulated and proven for braid-like tangles (tangles with a fixed number of bottom and top boundary points) as well: Just close the bottom/top of the tangle in all possible ways (one needs a bigger m for this) and proceed as above. This realizes the complex as bi-modules/bi-module maps over $\check{R}(\Lambda)$ as in the original formulation of Khovanov for his arc algebra, see [35].

A good question would be to extend Lemma 5.30 to braid-like tangles by checking the braid tangle moves (see for example Lemma X.3.5 in [32]) in our set-up.

5.2.5. *The calculation algorithm.* We now define an algorithm to compute the local differentials (that is, the ones from one resolution to another) of the complex $\mathrm{KR}(L_D)^n_{\mathrm{F}}$ using the HM-basis, and thus, the higher *Specht combinatorics*. We start by simplifying the notation: Since the canonical sequence from Definition 5.34 is fixed by (n^ℓ) and therefore by our presentation of the link diagram using q-skew Howe duality, we *suppress* to write $\hom_{\check{R}}(F^c_{(n^\ell)}, \cdot)$ in the following.

Example 5.40. Let us give the complex associated to Hopf link from Example 4.33 as an example. Recall that we have colored it with 1 and 2 and the presentation via $F_i^{(j)}$'s was

$$\mathrm{Hopf} = F_4^{(3)}F_5^{(2)}F_3^{(2)}F_2^{(2)}F_1^{(2)}T_{2,1,3}T_{1,2,2}F_5F_4F_3F_1F_2^{(3)}v_{(3^2)}.$$

Let us shortly write F_t and F_b for the string of $F_i^{(j)}$'s after (at the top) and before (at the bottom) the crossings $T_{2,1,3}T_{1,2,2}$ and v for $v_{(3^2)}$. Then the chain complex associated to it is, in simplified

notation, given by

$$\begin{tikzcd}[column sep=tiny]
 & F_tF_4^{(2)}F_3^{(2)}F_2F_3F_bv\{-1\} \arrow[dr, ": F_2F_3\to F_3F_2"] & \\
F_tF_4F_3^{(2)}F_4F_2F_3F_bv\{-2\} \arrow[ur, ": F_4F_3^{(2)}F_4\to F_4^{(2)}F_3^{(2)}"] \arrow[dr, ": F_2F_3\to F_3F_2"'] & \oplus & F_tF_4^{(2)}F_3^{(2)}F_3F_2F_bv\{0\} \\
 & F_tF_4F_3^{(2)}F_4F_3F_2F_bv\{-1\} \arrow[ur, "-: F_4F_3^{(2)}F_4\to F_4^{(2)}F_3^{(2)}"'] &
\end{tikzcd}$$

with leftmost part in homological degree zero. Moreover, there is no extra shift for the q-degree.

We point out that every step in the following definition is given by an *algorithm*.

Definition 5.41. (**Computation algorithm**) Given a oriented, colored link diagram L_D with cr crossings $c_{a,b}$ and a fixed presentation of it using q-skew Howe duality

$$L_D = \prod_k F_{i_k}^{(j_k)} \cdot T_{i_{\mathrm{cr}}}^{\pm} \cdots T_{i_1}^{\pm} \cdot \prod_k F_{i_k}^{(j_k)} v_{(n^\ell)} \tag{5.2.1}$$

with $T^{\pm}$'s for the or (as before, we allow extra F's between the different $T^{\pm}$'s), we assign to it a complex $\mathrm{KR}(L_D)^n_{\mathrm{F}}$ as in Definition 5.31.

Fix two vertices v_1, v_2 in the Khovanov cube associated to L_D that are connected by an edge and assume that v_1 is in lower homological degree. For both vertices we have a string of $F_i^{(j)}$'s associated to it that we denote by F_{v_1}, F_{v_2}. We also denote the associated $\check{R}(\Lambda)$-modules by M_1, M_2. Then there is local differential $d\colon M_1 \to M_2$ of the form as in Definition 5.28.

Then the local differential $d\colon M_1 \to M_2$ of $\mathrm{KR}(L_D)^n_{\mathrm{F}}$ can be computed in the following way.

- Compute the thick HM-basis for M_1 that we have defined in Definition 5.10. Denote the elements of this basis by $m_1^1, \dots, m_1^{k_1}$. These elements are given by string diagrams from $F^c_{(n^\ell)}$ at the bottom to F_{v_1} at the top.
- Compute the dual thick HM-basis for M_2 that we have "defined" in Remark 5.21. Denote the elements of this basis by $m_2^1, \dots, m_2^{k_2}$. These elements are given by string diagrams from F_{v_2} at the bottom to $F^c_{(n^\ell)}$ at the top.
- The differential d is a diagram with F_{v_1} at the bottom and F_{v_2} at the top.
- Thus, the composition $m_2^{k_{r'}} \circ d \circ m_1^{k_r}$ for each pair r, r' is $d_{rr'} \in \hom_{\check{R}}(F^c_{(n^\ell)}, F^c_{(n^\ell)})$.
- Define a matrix $d = (d_{rr'})$ consisting of these $d_{rr'}$ for $r = 1 \dots, k_1$ and $r' = 1, \dots, k_2$ scaled by the values that come from pairing the duals $m_2^1, \dots, m_2^{k_2}$ with the usual basis.

Theorem 5.42. *The algorithm of 5.41 gives a way to compute the homology of* $\mathrm{KR}(L_D)^n_{\mathrm{F}}$ *and thus, the colored Khovanov-Rozansky* $\mathfrak{sl}_n$*-link homology* $\mathrm{KR}(L_D)^n$.

Proof. To simplify notation: Let us denote by $\widehat{\cdot}$ the associated matrix factorizations (for strings of $F_i^{(j)}$'s) or homomorphisms of matrix factorization (for $\check{R}(\Lambda)$-diagrams) using Theorem 5.16.

First we note that we can use the local differentials from Definition 5.41 to define the differentials of $\mathrm{KR}(L_D)^n_{\mathrm{F}}$ by taking sums as usual if the local differentials of the algorithm coincide with the local ones from $\mathrm{KR}(L_D)^n_{\mathrm{F}}$. Then, by Theorem 5.37, we see that the complexes will have the same homology. The rest is linear algebra: Compute the kernels and images of the matrices, keep track of the gradings and obtain this way the homology of $\mathrm{KR}(L_D)^n_{\mathrm{F}}$. Hence, we have to ensure that the local differential agree. But this is also linear algebra:

- The two $\bar{\mathbb{Q}}$-vector spaces M_1 and M_2 are $\check{R}(\Lambda) - \check{R}(\Lambda)$-bimodules. Here the action from left (or right) is given by multiplying from the bottom (or top) by pre-(or post-)composing.
- Thus, by Theorem 5.16, they are also $H_n(\Lambda) - H_n(\Lambda)$-bimodules and the action is given in the same way. We see this way that $\hom_{\check{R}}(F^c_{(n^\ell)}, F^c_{(n^\ell)})$ is one dimensional and the $d_{rr'}$'s can therefore be seen as elements of $\bar{\mathbb{Q}}$ by choosing the "obvious" basis of the diagram that only points upwards.
- The local differentials from Definition 5.28 are exactly given by composing the corresponding $\widehat{d}$ to the left. Hence, $\widehat{d} \circ \widehat{m}^r_1$ is an element of $\mathrm{EXT}(\widehat{F}^c_{(n^\ell)}, \widehat{F}_2)$.
- Since the thick HM-basis is a basis that works in this generality, see Theorem 5.14, one can re-write $\widehat{d} \circ \widehat{m}^r_1$ in terms of the basis for $\widehat{M_2}$.
- But using the dual basis as in Definition 5.41 as above is nothing else then using the trace that we have recalled in Definition 3.33. This is nothing else than taking the inner product $\langle \widehat{d} \circ \widehat{m}^r_1, \widehat{m}^{r'}_2 \rangle$. Thus, the $d_{rr'}$'s count the multiplicity of $\widehat{m}^{r'}_2$ if one re-writes $\widehat{d} \circ \widehat{m}^r_1$ in terms of the thick HM-basis for $\widehat{M}$ (and scales the result as above).

Thus, we obtain the statement by Theorem 5.37. □

Example 5.43. Let us give a small but hopefully illustrating example how the calculation works. This is based on Example 4.32 from before. We note that we cheat again, since, if we would strictly follow the algorithm, then we would have to write U_D using a longer string of $F_i^{(j)}$'s.

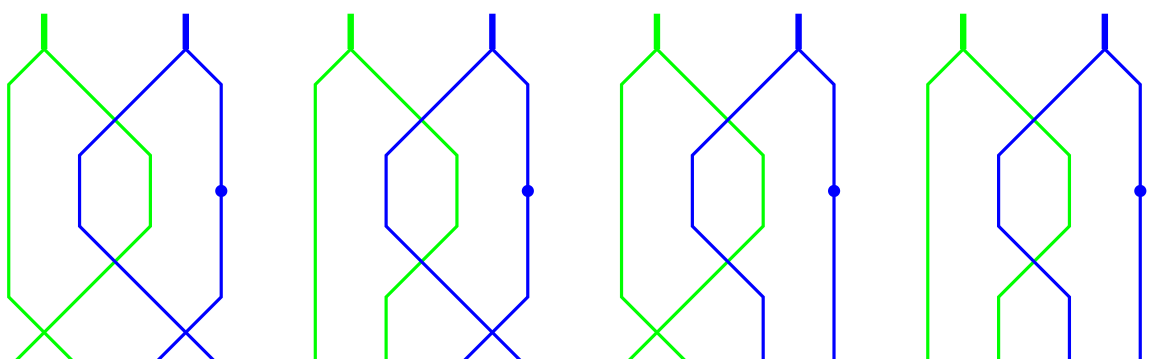
We write just $v = v_{(2^1)}$. We get the following chain complex for the diagram of the unknot U_D.

$$F_2 F_1 F_2 F_1 v\{-2\} \xrightarrow{\times :\, F_1F_2 \to F_2F_1} F_2 F_2 F_1 F_1 v\{-1\}$$

Here the right part is homology degree zero. Thus, we need to calculate the thick HM-basis for $\hom_{\check{R}}(F_2^{(2)} F_1^{(2)}, F_2F_1F_2F_1)$ and the dual one for $\hom_{\check{R}}(F_2F_2F_1F_1, F_2^{(2)}F_1^{(2)})$. We have already done the first in Example 5.22.

Note now that the 2-multitableaux for $F_2^{(2)}F_1^{(2)}$ is still $\vec{T}$ from Example 5.22. Moreover, we have four for $F_2F_2F_1F_1$, namely $\vec{T}_{1,2}$ and $\vec{T}_{3,4}$ from Example 4.14. Recall that the dot placement is just given by the associated dual standard filling $T^*_{\vec{\lambda}}$ where $\vec{\lambda}$ is the shape of the $\vec{T}$'s.

From this we get a sequence of transpositions τ from $\vec{T}_k$ to $T^*_{\vec{\lambda}}$. For the first two 2-multitableaux we have $\tau_2(1,2)\tau_3(2,2)\tau_1(1,1)\vec{T}_1 = T^*_{\vec{\lambda}}$ and $\tau_2(1,2)\tau_1(1,1)\vec{T}_2 = T^*_{\vec{\lambda}}$ and $\tau_2(1,2)\tau_3(2,2)\vec{T}_3 = T^*_{\vec{\lambda}}$ and $\tau_2(1,2)\vec{T}_4 = T^*_{\vec{\lambda}}$ for the last two. Thus, we have the four dual basis elements

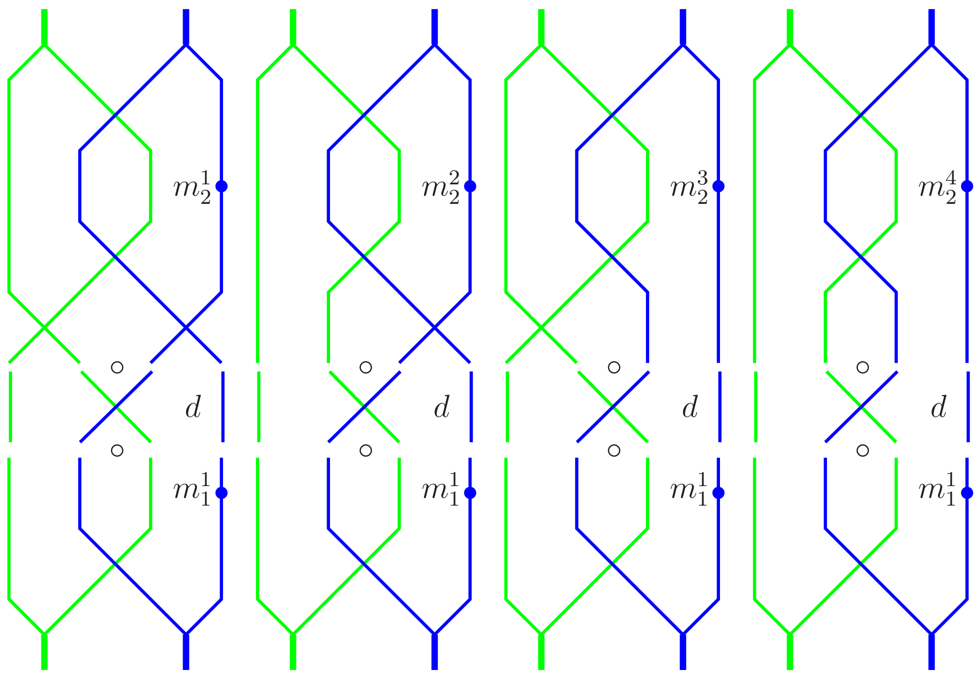


$\vec{T}_1 = (\boxed{1\,|\,3}\,,\,\boxed{2\,|\,4})$ $\vec{T}_2 = (\boxed{1\,|\,4}\,,\,\boxed{2\,|\,3})$ $\vec{T}_3 = (\boxed{2\,|\,3}\,,\,\boxed{1\,|\,4})$ $\vec{T}_4 = (\boxed{2\,|\,4}\,,\,\boxed{1\,|\,3})$

Applying the isomorphism to the $\mathfrak{sl}_2$-web algebra (and cheating again using Bar-Natan's cobordisms as in Example 5.24) we see that these corresponds from right to left to a pair of undotted cap's, a pair of cap's where one has a dot and a pair of cap's where both have a dot. To make connections to Definition 5.41, let us denote them by m_2^1, m_2^2, m_2^3 and m_2^4.

Moreover, the basis of the source from Example 5.22 can be read as a cup with a dot (denoted by m_1^1) and an undotted cup (denoted by m_1^2) and the differential d is the usual comultiplication. Thus, we expect that $d \circ m_1^1$ will pair with everything except one element of the dual basis to zero.

So let us evaluate the pictures which are just given by stacking now. We have



Note that it is exactly as we expected: All of the diagrams above give a $\bar{\mathbb{Q}}$ multiple of the trivial diagram with only two upwards pointing thick strands. And all with the exception of the left one are zero. To see this note that the rightmost two diagrams are on the nose zero because of two dots on the same strand (we are in $n = 2$). The second is zero which can be deduced from the thick calculus rules (see e.g. [42] or [69]). That is, opening the bottom Reidemeister 2 moves gives two terms: $\pm$ one with a dot on the green (left) strand $\mp$ one with a dot on the blue (right) strand. The second term is always zero, since the middle crossing is a composition of a split$\circ$merge. Thus, at the bottom we have a merge$\circ$split with two dots - this is always zero for $n = 2$.

But the same holds for the top now: Only a dot on the green (left) strand can survive after opening the Reidemeister 2 move. But then we have two dots on the green (left) strand which is zero in $n = 2$. Thus, the whole composition is zero.

The first one on the other hand gives ± 1: Only one term survives the opening of the Reidemeister 2 moves and it has exactly one dot between each merge$\circ$split-pair. Thus, they can be reduced to a line (up to a sign), see e.g. Corollary 2.4.2 in [42]. This shows that $d(m_1^1) = \pm m_2^1$.

Doing the same for m_1^2 (which has two surviving, namely m_2^2 and m_2^3) we see that d is given (up to a sign) by Khovanov's original comultiplication map which comes from the algebra $\bar{\mathbb{Q}}[X]/X^2$, see [34], namely $1 \mapsto 1 \otimes X + X \otimes 1$ and $X \mapsto X \otimes X$. This map is injective which shows that the homology is trivial.

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